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COLLECTIONS OF QUESTIONS,

PROPOSED AND RESOLVED BY

INGENIOUS CORRESPONDENTS.

IN QUARTERLY NUMBERS.

CONDUCTED BY

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PREFACE.

THE principal object of the present little work is to excite the genius and industry of those who have a taste for mathematical studies, by affording them an opportunity of laying their speculations before the public in an advantageous manner; and thus to spread the knowledge of mathematics in a way that is both effectual and agreeable. It is well known to mathematicians, that nothing contributes more to the development of mathematical genius, than the efforts made by the student to discover the solutions of new and interesting questions; and accordingly we find that many attempts have been made to apply this fact in such a manner as to render it most beneficial to society.

With this view, many periodical works, embracing mathematical inquiries, have been published in Great Britain: as, The Ladies' Diary, The Gentlemen's Diary, The Mathematical Companion, Dr. Hutton's Miscellanea Curiosa, Leybourn's Mathematical Repository, &c. These publications have had great influence on the state of mathematical science in that country; and, according to the opinion of some persons well acquainted with this subject, have advanced the knowledge of mathematics more rapidly and extensively than many other works of greater magnitude. Indeed, there is scarcely any thing that can give a better view of the very general diffusion of mathematical knowledge in Great Britain, than an examination of the works which we have just mentioned.

The English Ladies' Diary, which is published annually, was begun in 1704, and has continued till the present time. Among its successive conductors, were the eminent mathematicians, Simpson and Dr. Hutton; and in the list of its contributors are enrolled the names of many of the best mathematicians that England has ever produced; it is sufficient to mention Emerson, Simpson, Landen, Lawson, Vince, Hutton, Dalby, Major Henry Watson, Wales, and Mudge. It contains a great number of interesting and useful problems in all the branches of mathematics.

Leybourn's Repository is a work of great merit. Many of its problems, solutions, and researches, are learned and ingenious.

It is supported by the labours of the first mathematicians in England, as Barlow, Ivory, Professor Wallace of Edinburgh, and many others of distinguished abilities.

Similar works have been long in use among the mathematicians of Ireland. The Ladies' Diary and the Belfast Almanac are publications of considerable utility in extending the knowledge of mathematical science. The latter is under the direction of Professor Thompson, of the Belfast Institution, a gentleman extremely well qualified for the work by his talents and impartiality.

In this country, also, several works of a similar kind have appeared, and have been productive of some advantages in improving, as well as in disseminating, the science of mathematics; but their usefulness has been limited by various causes, which, it is hoped, will not operate on the present undertaking. These works, as they successively appeared, were, The Mathematical Correspondent, conducted by the late ingenious Mr. George Baron; The Analyst, by the Editor of the present work; The Scientific Journal, by Mr. Marrat; and the Philosophic Magazine, or Gentleman's Diary, by Mr. Nash. Among the contributors to these publications, were several ingenious and learned mathematicians; as, Gummere, the author of two good elementary treatises, the one on surveying, the other on astronomy; the very ingenious and much lamented Professor Fisher; and the profound mathematician, Dr. Bowditch.

In the present work, which will be published in quarterly numbers, contributors of new discoveries or improvements in mathematics, or of new problems and solutions, shall have their communications published with accuracy, and ascribed to their respective authors.

A prize question will be proposed in each number of the work; and the choice of the question will depend on its elegance, curiosity, or utility, in improving or extending science. For the best solution, a prize will be given of ten copies of the number containing the solution; and that number will be designated by the name of the person who obtains the prize.

THE

MATHEMATICAL DIARY.

ARTICLE I.

ESSAY

ON THE RECTIFICATION AND QUADRATURE OF THE CIRCLE.

1. The mensuration of the circle is a subject which has occupied the attention of mathematicians in all ages; and all the resources of their science have been employed, in order to obtain its measure with the greatest degree of exactness. The astonishing perseverance with which the numerical values of the area and circumference have been calculated, can only be equalled by the ingenuity which has been displayed in discovering such methods of calculation as would produce the required values with the greatest facility. Great numbers, even of those who were unable to comprehend the profound investigations of mathematicians, have devoted themselves to the study of this beautiful, and apparently simple figure, with the greatest diligence: some of whom, under the influence of a strong imagination, have fancied themselves in possession of complete solutions to the most difficult and important problems respecting the circle, which the most ingenious and learned geometers had never been able completely to unfold. The labour, however, which has been bestowed on this subject, did not proceed from curiosity alone: the utility of the circle in all parts of the practice, as well as in the theory of mathematics, accounts, in a great measure, for the industry with which this part of geometry has been cultivated. We have only to reflect on the application of the circle to the mensuration of angles in trigonometry, both plane and spherical, to modern analysis, and consequently to nearly all the grand problems in physical astronomy, to practical astronomy, geography, and navigation, and we shall be fully convinced, that few things in the elements of mathematics have greater claims upon our attention than the figure at present under our consideration.

2. It appears, from the writings of Euclid, Archimedes, and others, that the elementary relations of various straight lines and rectilineal figures connected with the circle, were well known to

the geometricians of ancient Greece ; and the moderns, in tracing the footsteps of their illustrious predecessors, have not failed to generalize many of their theorems, as well as to add great numbers of new properties of the circle which were unknown to the ancients. But of all the problems respecting the circle, which have engaged the attention of mathematicians, its rectification and quadrature have been sought with the greatest avidity. The rectification of the circle consists in determining the circumference, when the diameter is given; or, more particularly, in finding a straight line equal to the circumference. The quadrature of the circle consists in determining the area, or extent of the surface of the circle, when the diameter is given; or, as the term indicates, in finding a square equal in magnitude to the circle. In the first of these problems, the object of research is a line; in the second, a surface. But notwithstanding this essential difference in the nature of the things required, it has been fully demonstrated, that the solution of either of these problems readily furnishes the solution of the other; and, therefore, the difficulty of investigation is the same for both.

3. There are two different points of view under which these problems may be contemplated. In the first of these views we consider the whole circle: the objects of research are, the whole circumference and the whole area, the determinations of which are called the definite rectification, and quadrature of the circle; and these are the values which have usually been sought by such as have undertaken to square the circle. The second of these views is general, and comprehends the determination of an arc of a circle, or of a segment, when we have the chord, sine, tangent or secant, &c. of the arc: this determination is called the indefinite rectification, or quadrature of the circle.

4. One of the simplest modes in which the rectification of the circle can be conceived or expressed, consists in assigning two integer numbers, exhibiting the ratio of the diameter to the circumference; or, which is the same in effect, that one of the integers being the value of the diameter, the other may be precisely the value of the circumference. This ratio is the more valuable, on account of its being applicable to all circles; for it has been shown by various writers, that the ratio of the diameter to the circumference in any one circle, is the same with the ratio of those quantities in any other circle. It is scarcely necessary to remark, that the expression of this ratio by two integers, is equally extensive with any that could possibly be expressed by means of fractions: so that, if the ratio of the diameter to the circumference can be expressed by means of numbers, it can certainly be expressed by integers; and if it cannot be expressed by two integers, it follows that there is no possible expression for it by means of numbers. This perfect numerical ratio of the diameter to the circumference has never been obtained; and the failure has given rise to many subtile speculations respecting the nature of the circumference. Some suppose that we have not expressed the ratio

with perfect accuracy, because our modes of calculation have been conducted in an imperfect manner; others, that our schemes of computation have been radically defective or inadequate; others, that the flexure or curvature of a line involves some incompatibility between its measure and that of a straight line, and that on this account the circumference of the circle is not susceptible of an exact numerical comparison with the diameter: lastly, some maintain, that although there is no necessary incongruity between two lines, of which one is straight and the other curve, there is, notwithstanding, in the nature of the circle, some peculiarity that renders the ratio of the circumference to the diameter inexpressible in numbers.

5. With respect to the effect of curvature, in rendering lines possessed of it numerically incompatible with straight lines, it is easy to prove that there is no foundation for such a supposition. There are many curve lines which are perfectly susceptible of comparison with straight lines: of these we have an example in the celebrated curve called the cycloid. In this curve, which is described by a point in the circumference of a circle, which revolves on a straight line and in one plane, as a nail in the wheel of a carriage, we can compare, in simple integer numbers, any arc of it between given points with certain given straight lines, which is its indefinite rectification: and in particular, the whole curvilineal track of the nail in space from its leaving the ground till its return to it again, is exactly equal to four times the diameter of the revolving wheel.

6. It is not in curve lines only, that mathematicians fail to obtain numerical relations to straight lines: they are frequently at a loss to determine the value in numbers of the ratio between two straight lines. When the base and perpendicular of a right angled triangle are given in numbers, and the hypothenuse is sought, we sometimes succeed, and at other times fail, according to the particular nature of the given numbers expressing the base and perpendicular. If we assume 3 and 4 for the sides about the right angle, the hypothenuse is found by adding the squares 9 and 16 of the given numbers, and extracting the square root of the sum 25: this root is 5, which is the length of the hypothenuse, which has to its base the exact ratio of the integers 5 and 3. Again: if the base and perpendicular be 5 and 12, we follow the same rule as before in squaring the given numbers, adding the squares, and extracting the square root of the sum: the square of the hypothenuse is $25+144=169$, the square root of which is 13, which is the exact length of the hypothenuse. Several other cases may be easily found, in which the length of the hypothenuse is obtained with perfect accuracy: but if we use any numbers at pleasure for the sides about the right angle, we shall seldom succeed in determining the hypothenuse with numerical exactness. Thus, if we assume 2 and 3 for the base and perpendicular, the square of the hypothenuse is $4+9=13$, and the hypothenuse, if it be capable of a numerical expression, must be the square root of 13. But we

know that there is no number, integral or fractional, that, multiplied by itself, produces the number 13; and, therefore, the hypothenuse to the legs 2 and 3 is inexpressible in numbers. If we choose 2 and 1 for the two legs, we have $1+4=5$ for the square of the hypothenuse; and as 5 has not an exact square root, therefore in this case also the hypothenuse has not a numerical value. From these examples, and others which it is easy to adduce, we perceive that the ratio of two straight lines may be incapable of an exact numerical expression: from which it seems reasonable to conjecture, that curve lines compared with those that are straight, may, in some cases, be subject to a similar deficiency in numerical proportion.

7. It is a problem of considerable interest and utility, to ascertain all the cases in which the base, perpendicular, and hypothenuse of a right angled triangle are explicable in numbers. The solution is obtained with the greatest simplicity in the following manner: Suppose we would derive the base from the hypothenuse and perpendicular; we must take the difference of the squares of these lines, which will be the square of the base, and must therefore be a square number: but the difference of the squares of two straight lines is equal to the rectangle of their sum and difference, which rectangle is expressed, numerically, by the product of the numbers that denote the sum and difference; this product, therefore, must be a square number. We may therefore assume any two numbers for the sum and difference, provided that the product be a square number; and having the sum of the hypothenuse and base, and also their difference, those sides will be found by addition and subtraction. Thus we may assume, for their sum and difference, the two numbers 8 and 2, of which the product is the square number 16: by adding and subtracting the numbers 8 and 2, we have 10 and 6; the halves of which, 5 and 3, are the hypothenuse and base, and the root of the square 16, namely, 4, is the perpendicular; and therefore the three sides are 3, 4, 5.

8. Or, we may begin with assuming any number whatever for the base, the square of which, divided by any assumed number, as the sum or difference, will give the other in the quotient, whence the hypothenuse and remaining side are found as before. To take an example of this rule, assume 6 as the base, of which the square is 36; now divide this square by the number 18, and the quotient is 2; and 18 and 2, being added and subtracted, give 20 and 16, the halves of which are 10 and 8; and therefore the sides of the triangle are 6, 8, 10.

These rules are general, and furnish all the cases that can exist; but they may be exhibited in a different form. Let b=the given base, and h and p the hypothenuse and perpendicular; then, assuming s for the sum, we have by the last rule $\frac{b^2}{s}$ for the difference, whence by addition and subtraction, we have $s+\frac{b^2}{s}$ and

$s-\frac{b^2}{s}$, of which the halves are $\frac{s}{2}+\frac{b^2}{2s}$ and $\frac{s}{2}-\frac{b^2}{2s}$, which are the hypothenuse h and the perpendicular p. If we reduce the three values of h, p, and b, to a common denominator, and reject the denominator, we have for the sides of the triangle,

$$s^2+b^2,\ s^2-b^2,\ 2bs.$$

These values of h, p, and b, are general, as far as regards the ratios of the sides; but in consequence of rejecting the common denominator, all the possible triangles having sides of different magnitudes, are not comprehended. Thus the three sides 15, 12, 9, which have the ratios of 5, 4, 3, are not included in the formulæ given in terms of b and s.

This remark shows us, that the rule given for this problem by some writers on the diophantine analysis is incorrect. Emerson, an English mathematician of considerable eminence, in his Algebra, page 238, second edition, has the following passage: "But if the answer is required in whole numbers, then $2rx$, $xx-rr$, $xx+rr$ will denote the roots of the squares, where the sum of the two first is equal to the last square."

Instead of using this theorem as given by the author, we must take an arbitrary multiplier m, and all the possible triangles having integral sides will be expressed by the formulæ,

$$m(s^2+b^2),\ m(s^2-b^2),\ 2mbs.$$

The algebraic methods of resolving this problem are to be found in several popular works of great merit, as Bonnycastle, Euler, &c. and therefore do not require our immediate attention.

9. It was well known to the ancient geometricians, Pythagoras, Plato, Euclid, and others, that various straight lines were incommensurable with each other, as appears from the tenth book of Euclid's Elements; and the same thing has been well discussed by modern writers, whose solutions it is unnecessary to exhibit at the present time. There is one case, however, which, on account of its singularity and simplicity, deserves a particular consideration. The case to which I have reference is that of a right angled triangle, in which the base and perpendicular are equal, and the ratio of the hypothenuse to either side is sought; which is evidently the same as the ratio of the diagonal of a square to its side.

If it were possible that the side of a square and its diagonal were both expressible in numbers, and consequently in integers, the ratio might be reduced to its least terms by dividing both the integers by their greatest common divisor; and since the numbers thus obtained cannot both be even, one of them must be an odd number. Now it is obvious that the diagonal cannot be an odd number, because the square of an odd number is also an odd number, and cannot be divided into two equal integer square numbers for the squares of the two equal sides. Again, the diagonal cannot be an even number; for if it were, its square would be an even number divisible by 4, for the square of every even number is well known to be divisible by 4; and consequently when the square of the hy-

pothenuse is divided into two equal parts for the squares of the equal sides, each of those parts will be an even number, and consequently the root must be even also; so that when the diagonal is even, the side must be even, which is impossible, because of the diagonal and side, one must be an odd number. Thus, it appears, that in expressing the ratio of the diagonal to the side in the least terms in integers, the diagonal belongs neither to the class of even nor of odd numbers, and therefore the ratio of the diagonal of a square to its side cannot be expressed in numbers.

10. From what has been said, it will not appear surprising to find the circumference of the circle incommensurable with its diameter; and this seems to be the reason that mathematicians have never succeeded in their attempts to give the ratio of the circumference by integer numbers. In this sense of the problem, it must be confessed that the rectification of the circle has never been found. Several persons, indeed, have imagined that they had discovered the precise numbers which give the ratio of the diameter to the circumference; but their pretensions have always been unsupported by satisfactory demonstrations, and their numbers have been found beyond those limits within which the true ratio must necessarily be confined. The celebrated Danish astronomer, Christian Longomontanus, who flourished about the year 1600, laid claim to the discovery of the exact numerical rectification of the circle, and maintained that the ratio of the diameter to the circumference was that of the two integers 100000 to 314185. His error was pointed out by Snellius, Briggs, Guldinus, and others, who showed him that the second term of his ratio ought to be between the numbers 314159 and 314160; and Dr. Pell, a learned English mathematician, proved clearly that the ratio assigned by the Danish astronomer made the circumference of the circle greater than the perimeter of the circumscribing polygon of 236 sides. Leistner, who was an officer in the French service a few years ago, also pretended that he had succeeded in discovering the exact numerical ratio of the diameter to the circumference; his ratio is that of 1225 to that of 3844. It is easy to prove that this ratio is erroneous, for it does not fall between the limits $3\frac{1}{7}$ and $3\frac{10}{71}$.

11. The mathematicians, having been disappointed in obtaining the exact numerical value of the circumference from the diameter, attempted the rectification by geometrical constructions. Of these, the simplest and the most valuable are performed by the Postulates adopted in the Euclidean geometry, that is, by the description of straight lines and circles. But all the efforts of geometricians have hitherto been ineffectual in reducing the rectification of the circle to constructions of this elementary kind. Some, it is true, have indulged themselves in the belief that they had found, by this method, a straight line equal to the circumference of the circle, among whom were the celebrated Joseph Scaliger, and Hobbes the English metaphysician; but their labours were altogether useless, as was demonstrated by Clavius, Vieta, Wallis, and others.

12. Having been unsuccessful in their endeavours to obtain a numerical or geometrical solution of the simplest kind, mathematicians attempted the rectification and quadrature of the circle, by the help of constructions derived from various curves; but no algebraic or geometrical curve has yet been found, that, by its intersections with straight lines, or with other curves of the kind denominated algebraic or geometrical, will determine the circumference of the circle. To prevent misconception, it may be necessary to observe, that by algebraic or geometrical curves, I mean those that have the relation of their rectangular co-ordinates expressed by any limited function of those co-ordinates, that is, by a formula of a finite number of terms, involving the abscissa with its corresponding rectangular co-ordinate. Of such curves, none has been found applicable to the problem before us: and thus, we have passed through three grades of science, without arriving at any principle or relation that exhibits the ratio of the diameter to the circumference, either in a geometrical or algebraical form.

13. If, however, we advance to the fourth grade of science, that is, to such curves as are denominated mechanical or transcendental, we find some that involve in their nature the ratio of the diameter and circumference of the circle. Of these, the earliest of which we have any account, is the quadratrix of Dinostrates. This ingenious mathematician, who was the pupil or friend of Plato, seems to have invented the curve which is called by his name, for the purpose of dividing the quadrant, or any given arc of a circle, in a given ratio. This section of a circular arc in a given ratio is effected with great simplicity and elegance by the curve in question. Besides this property, however, it possesses another, which is immediately to our purpose: by its intersection with a radius of the circle, it cuts off from the radius a segment which has the same ratio to the radius, that the radius has to the quadrantal arc of the circle. Nothing is simpler, or more ingenious in theory, than this determination of the two straight lines which measure the circle; but with respect to its practical application, several inconveniences or imperfections are found to attend it.

In the first place, supposing the curve to be actually described, no method of calculation appears to result from it, by which the circumference can be determined in numbers; so that in this respect, the quadratrix has been of no utility. Again, the several points of the curve being determined by the intersection of two straight lines, one of which moves with a uniform motion in one plane about a given point as a centre, and the other with a uniform motion parallel to itself in the same plane, the point of the curve which gives the rectification of the circle is found only when the two straight lines coincide; but when two lines coincide, there is, strictly speaking, no intersection, and therefore the point of the quadratrix, that is necessary for determining the circumference of the circle, is not found in a perfect manner. If any one point of two coincident straight lines may be taken for a point of intersection, then every point of the lines is equally so; and thus the straight line that bisects

the curve at right angles, and which is properly the axis of the figure, must be reckoned a part of the line denominated quadratrix: or if coincident straight lines cannot have an intersection, then the point in which the curve meets the axis is not obtained directly by construction, but must be inferred from the nature of the curve where it is immediately proximate to the point required: and in this case the position of the point sought is determined simply by the assumption of the law of continuity, that is, on the hypothesis that the point sought may be approximated indefinitely by points in the curve, which are accurately determined by construction.

14. These observations will be found to be perfectly consistent with the results that we obtain from the equation of the curve. Let x and y be the rectangular co-ordinates of any point of the curve, both being reckoned from the centre of the generating circle, the latter on the axis towards the vertex of the curve which is to be ascertained, and the former on a straight line at right angles to the axis: also let z=the angle at the centre, contained between the axis and the straight line drawn from the centre to the point denoted by x and y, this angle being measured on the primitive circle, the radius of which is denoted by unity; and let π be the semicircumference of the same circle.

By the definition of the curve, and by similar triangles, we have

$$z=\frac{\pi x}{2}, \text{ and } \frac{x}{y}=\text{tang. } z.$$

Eliminating x from these two equations, we have the equation

$$\frac{\pi}{2}\cdot y=\frac{z}{\text{tang. } z}.$$

Now, the quantity sought being the distance from the centre to the vertex of the curve, we have only to determine y when z vanishes, and this value of y will be the distance required. We have therefore to substitute 0 for z in the preceding value of y, and

$$\frac{\pi}{2}\cdot y=\frac{0}{0}.$$

This expression, it is evident, is perfectly equivalent in signification to the result obtained, by supposing the intersecting lines to coincide: it shows that the value of y is absolutely indeterminate, or that every part of the axis may be considered as the extremity of y.

But if we would find that point which is connected with the curve by the law of continuity, we have only to substitute for tang. z its value in terms of z, which is

$$\text{tang. } z=z+\frac{z^3}{3}+, \text{ \&c.}$$

and the general value of y in terms of z, is converted into

$$\frac{\pi}{2}\cdot y=\frac{z}{z+\frac{1}{3}z^3+, \text{ \&c.}}=\frac{1}{1+\frac{1}{3}z^2+, \text{ \&c.}}$$

which, when z=0, becomes

$$\frac{\pi}{2}\cdot y=1, \text{ whence } y=\frac{1}{\frac{1}{2}\pi}=\frac{2}{\pi}.$$

From this equation we easily derive the stating

$$\frac{\pi}{2} : 1 :: 1 : y,$$

which value of y agrees with the distance of the principal vertex from the centre, as determined by Emerson, in his Account of Curve Lines, and Bonnycastle in his Algebra, and others.

If we put $t =$ the tang. z, we have, by a known series,

$$z = t - \frac{t^3}{3} + \frac{t^5}{5} -, \text{ \&c.};$$

and substituting these values of z and tang. z, we have y by the equation

$$\frac{\pi}{2} \cdot y = \frac{t - \frac{t^3}{3} + \frac{t^5}{5} -, \text{ \&c.}}{t} = 1 - \frac{t^2}{3} + \frac{t^4}{5} -, \text{ \&c.};$$

and putting $t=0$ in this equation, it is reduced to the same equation as before,

$$\frac{\pi}{2} \cdot y = 1.$$

We might also proceed by using the equation of x and y, which, by eliminating z from the two primitive equations, is found to be

$$y = \frac{x}{\text{tang. } \frac{\pi x}{2}};$$

from which, by proceeding nearly as in the preceding calculation, we obtain the same result as before.

The practical operations by which the quadratrix may be described, are not of such a nature as to be susceptible of great accuracy, as will be evident, if we consider that both the generating motions must be uniform; or, as an equivalent, one of the motions must produce the other, so that a rotatory motion must produce a rectilineal motion or reciprocally, in such a manner that the two motions shall have an invariable ratio. We may conclude, therefore, that even admitting the theoretical perfection of the quadratrix of Dinostrates, in exhibiting the ratio of the circumference to the diameter by means of two straight lines, it is of little or no service in obtaining a practical rectification of the circle.

It does not appear that the inventor of this curve, or the mathematicians in general who described it since his time, have conceived the quadratrix in its full extent. One infinite branch, beginning at the principal vertex, and extending indefinitely along an asymptote, is all that is commonly assigned to the curve. The only writer who has given a complete idea of this extraordinary figure is Leslie, the celebrated Scottish philosopher, who has shown that the whole curve consists of an infinite number of branches, extending indefinitely both ways along an infinite number of asymptotes parallel to the axis, and distant from each other by an interval equal to the diameter of the generating circle; and as these branches and asymptotes succeed each other without limit

both ways from the axis, the quadratrix may be conceived to occupy the whole of infinite plane space.

15. But notwithstanding the extent of the quadratrix of Dinostrates, it is only a particular case of a more general curve, described by the intersection of two straight lines revolving in the same plane, with given angular velocities about two centres, one of which is fixed, and the other is supposed to move in a straight line passing through the fixed centre, and at right angles to the straight line in which the revolving lines coincide.

The analytical expressions which express the nature of this curve, are obtained with great facility. Let z and mz be the measures to the radius unity of the angles described about the fixed and variable centres from the line of coincidence, nz = the uniform velocity of the moving centre, and x and y the rectangular co-ordinates of any point in the curve reckoned from the fixed centre, the latter on the line of coincidence, and the former on the line of centres: also let A and B be the distances from the fixed centre to the points in which the curve meets the line of centres and the line of coincidence, which is the axis of the curve. The primitive equations derived from the curve, are easily perceived to be

$$\frac{x}{y}=\text{tan. } z, \text{ and } y \text{ tan. } z-y \text{ tan. } mz=nx.$$

From these equations we have the following values of x and y:

$$x=\frac{nz \text{ tan. } z}{\text{tan. } z-\text{tan. } mz}, \quad y=\frac{nz}{\text{tan. } z-\text{tan. } mz};$$

by means of which, all questions respecting the curve may be determined. We shall only notice the relation of the values A and B. Let $z=\frac{\pi}{2}$=the measure of a right angle; then from the preceding equations for x and y, we have

$$A=x=\frac{n\pi}{2}, \text{ and } y=0.$$

Again: let $z=0$, and by the same equations for x and y, we have

$$x=0, \text{ and } B=y=\frac{n}{1-m}.$$

Lastly; eliminating n from the equations involving A and B, we have

$$\pi=\frac{2}{1-m}\cdot\frac{A}{B},$$

which gives the value of the quantity $\pi=3.14159$, &c. in terms of the given ratio m, and the given straight lines of the figure A and B. If $m=0$, we have the curve of Dinostrates, in which $\pi=2.\frac{A}{B}$.

16. The spiral of Archimedes is another curve, by the description of which the rectification of the circle may be obtained. Conceive an indefinite straight line to revolve in one plane about a

fixed point as a centre, with any given angular velocity measured on the circumference of a circle described about the same centre at the distance r; and suppose a point to move uniformly from the centre along the revolving line, with a velocity equal to that with which a point in the revolving line moves along the circumference of the circle to the radius r, and thus describe the spiral of Archimedes. Now, according to this combination of motions, it is obvious that the segment of the revolving straight line described in any given time by the point which moves along it, is equal to the corresponding arc of the circle described by the point of the revolving line; so that we have a straight line equal to any proposed circular arc, that is, we have the indefinite rectification of the circle.

Let $\rho =$ any variable distance from the centre described by the point that moves along the revolving straight line, and $z =$ the simultaneous arc described on the circumference to the radius r; and by the nature of the construction, we have

$$\rho = z,$$

which is the polar equation of the curve described by the point in the moving line. Now let $\pi =$ the semicircumference to the radius unity; and when the revolving line has made one revolution, the arc described is $2\pi r$: also let $C =$ the value of ρ at the end of this revolution, and we have

$$C = 2\pi r, \text{ whence } 2r : C :: 1 : \pi;$$

that is, the ratio of 1 to π, which is the ratio of the diameter of a circle to its circumference, is given, being equal to the ratio of the given straight lines $2r$ and C.

This curve has been described mechanically by several ingenious persons;* but the first and simplest method was discovered by the celebrated French geometer Clairaut, and published in the Memoirs of the Academy of Sciences for the year 1740. The description of this curve by Clairaut is exceedingly simple and ingenious, perfectly satisfactory in theory, and easily reduced to practice. An account of his method, therefore, will probably be acceptable to the reader.

Draw a straight line on a given plane, and also mark on the plane a point at any given perpendicular distance from the straight line: take a circular plane, of which the radius is equal to the perpendicular distance just mentioned, and place this circular plane on the given plane, so that its centre may coincide with the given point, and consequently its circumference will touch the given straight line. Now let the circular plane roll along the given straight line from the forementioned position, and the fixed point on the given plane will describe, on the moving plane, the spiral of Archimedes.

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* In the month of June last, Mr. A. Quinby, Teacher in the city of New-York, presented to Columbia College, a machine for the mechanical description of the spiral of Archimedes.

In order that the spiral may be described to any indefinite extent, we have only to conceive that the plane of the revolving circle is extended indefinitely, while the same circle rolls on the same straight line.

17. The cycloid is another curve, by the description of which the rectification of the circle is determined. Its manner of production is easily understood, by contemplating the motion of a nail in a carriage wheel. When the nail is on the ground, it begins to ascend from a state of momentary rest, and its velocity increases continually until it is directly above the centre of the wheel, in which position its velocity is greatest, and exactly double the velocity of the centre, its motion at that point being horizontal. The nail, in its descent, loses its velocity by the same degrees with which the velocity was acquired in the ascent; so that the descending branch of the curve is exactly equal and similar to the ascending branch, the whole line forming a graceful arch, of which the extremities are at the points where the nail touches the ground. By the continuance of the motion of the carriage, the nail describes a series of these arches, which are all equal and similar to each other, and resemble the arches of a bridge. The height of each arch is the diameter of the wheel, and the base of the arch is the straight line, along which the wheel revolves between two successive points, in which the nail touches the ground.

The base of the arch being described during one revolution of the wheel, is evidently equal to the circumference of the wheel; and thus the base and height of one of the cycloidal arches are the circumference and diameter of the same circle. Nothing is simpler in the theory, therefore, than the determination of the ratio of the diameter of a circle to its circumference, by means of the cycloid. We have only to make a circle of a known diameter roll in a plane along a straight line, until the circle has made one complete revolution; that is, until a point in the circumference of the revolving circle, setting out from the straight line on which the circle rolls, has returned to the same straight line again. It will be evident to any one who will compare the description of the cycloid with that of the spiral of Archimedes, as given by Clairaut, that these two methods are essentially the same, and are, in all probability, as simple, easy, and accurate, as any that can be found by the description of curve lines. It must, however, be acknowledged, that such constructions by curve lines, although beautiful in speculation, are of little comparative use in promoting the interests of mathematical science. The numerical ratio of the diameter to the circumference, cannot be determined in this way to a degree of accuracy required in the present state of the arts and sciences. And if constructions by geometrical curves could be obtained, the same difficulty with respect to the accuracy of the results would still remain. Even the intersections of straight lines and circular arcs would have very little advantage in this respect. We must, therefore, have recourse to more powerful methods of investigation, if we would attain to a high degree of exactness in the rectification and quadrature of the circle.

18. Mathematicians having sought, without success, for the exact ratio of the diameter to the circumference in integer numbers, have been under the necessity of employing approximations, by means of which the required ratio may be found to a degree of accuracy sufficient to answer the demands of those who may be engaged in calculations relative to the various branches of mathematical science.

Approximations are naturally divided into two kinds, limited and unlimited approximations. In the first of these kinds, a particular number is obtained by calculation from a certain geometrical construction; and this number once obtained, the method leads to no new value approaching nearer to the ratio required.

The unlimited, or indefinite approximation, is of more value, and furnishes results successively nearer and nearer to the quantity sought; so that the error may be rendered smaller than any given number. Such, indeed, is the advantage of this species of approximation, and so numerous the forms by which it may be conducted, that it may be considered as one of the most extensive and valuable means of research that have ever been discovered by mathematicians.

19. In making use of the limited approximation, we select the following example:

GEOMETRICAL CONSTRUCTION

For determining a straight line nearly equal to the quadrantal arc of a given circle.

In the given circle inscribe an isosceles triangle on a base equal to the given radius of the circle, and through the centre of the circle draw a diameter parallel to the base, and cutting off from the former another isosceles triangle; then will one of the equal sides and the base of the latter triangle in one sum be nearly equal to the quadrant of the circle.

To calculate the ratio of the circumference to the diameter from this construction, let the radius of the circle be taken equal to unity, which will also be the base of the first isosceles triangle. The altitude of this triangle consists of two parts; one of which is the altitude of the less triangle or unity, and the other part is the altitude of an equilateral triangle, having its base equal to unity, which altitude is evidently $\frac{1}{2}\sqrt{3}$; and therefore the whole altitude of the first isosceles triangle is $1+\frac{1}{2}\sqrt{3}$. Again; the semi-base of this triangle being $\frac{1}{2}$, the square of one of its equal sides is easily found to be $2+\sqrt{3}$, $=3.7320508$, whence, by extracting the root, this side $=1.9318517$, and therefore the sum of the base and one side of the large triangle is 2.9318517. Now say, by similar triangles, as the perpendicular of the greater triangle is to that of the less, so is the sum just found to the sum required, which is $=1.57117$, and doubled, is $=3.14234$; or, in five figures, 3.1423, the correct number to five places being 3.1416; and, therefore, the error is about the 4500th part of the true value.

(*To be continued.*)

ARTICLE II.

NOTICES OF BOOKS.

Elements of the Theory of Mechanics. By Guiseppe Venturoli, Professor of Mathematics in the University of Bologna. Translated from the Italian, by D. Cresswell, M. A., Fellow of Trinity College, Cambridge. To which is added, a Selection of Problems in Mechanics. Vol. I. 8vo. Cambridge. 1822.

Also, *Elements of Practical Mechanics.* Vol. II. 8vo. By the same author; published at Cambridge, 1823.

In the first volume, the author teaches the theory of mechanics in a systematic and elementary manner, but perhaps a little too concisely for beginners. He very properly divides his subject into two parts, Statics and Dynamics. In the statics are found most elementary subjects that belong to the science; among which we observe the composition of forces—centre of gravity—equilibrium of forces acting on one point—momentum of rotation—equilibrium of a system of invariable form—pressure upon the supports of a rigid system in a state of equilibrium—equilibrium of a rigid system acted on by parallel forces—systems of variable form—the funicular polygon—funicular curve—catenary and elastic curve. These articles are generally discussed by Venturoli, in a neat, accurate, and perspicuous manner, but somewhat too succinctly for such as have not previously employed some time in perusing other elementary works on the same subjects.

The author establishes his system of statics on the composition of forces. Having, in his first chapter, given the necessary definitions, he lays down the fundamental theorem of the composition of forces in the second proposition of the second chapter, which is as follows;

"Proposition II. If two forces act upon a point at any angle, and if the parallelogram, having for its sides the lines which represent them, be completed, its diagonal shall represent the resultant."

The demonstration given to this most important proposition, is of the same nature with that given by Newton, in the first Book of the Principia to his first Corollary to the laws of Motion, but without the advantage of any previous laws of motion from which the demonstration may be deduced; and on this account learners must experience considerable difficulty in obtaining adequate ideas of the subject. Venturoli takes no notice whatever of the demonstrations which have been given to this theorem, since the days of Newton, by Bernoulli, D'Alembert, Robison, Gregory, Poisson, Duchayla, and Laplace: all of which are independent of time and velocity, and may be justly ranked among the most valuable

improvements that have been made in the elements of mechanical science since the days of Newton.

In his fourth chapter, the author investigates the fundamental property of the lever with great simplicity and elegance. His method, which has long been familiar to mathematicians, consists in conceiving two equal additional forces applied at the extremities of the lever, and in the direction of it, but towards contrary parts. By means of these new forces, which do not disturb the preceding equilibrium, the parallel forces are converted into such as are oblique, and which, therefore, admit the application of the rule for the composition of forces.

In the sixth chapter we find the investigation of the centre of parallel forces in a general manner, in which he gives the beautiful theorem of La Grange, for determining the position of the centre of gravity of a system, when the masses of the bodies or particles of the system are given, and the several distances of those particles from each other. This theorem is demonstrated in the Analytical Mechanics of La Grange, vol. I. pages 64, 65, 66, of the second edition. The same theorem is given by La Place, in the third chapter of the first Book of the Celestial Mechanics.

In this sixth chapter also, Venturoli demonstrates the singular theorem relating to the indetermination of the pressures sustained by four or more points or props supporting an horizontal plane, on which a given weight is placed. This theorem is taken notice of by Playfair, in his admirable compend entitled, "Outlines of Natural Philosophy."

In the eleventh chapter of the Statics, Venturoli teaches the theory of Guldinus, or "the use of the centre of gravity in measuring surfaces and solids generated by revolution." The following is his third proposition on this subject:

"If a straight line move so as to remain always perpendicular to the line described by its centre of gravity, the surface generated by the motion of the straight line is equal to the straight line itself multiplied by the path of its centre of gravity; and the same is true in the case of a plane, which, moving in like manner, generates a solid."

There is no reason to doubt the truth of this theorem, when taken in the sense intended by the author. When the generating quantity is a straight line, and the path of its centre of gravity, or middle of the line, a plane polygon or plane curve, the surface contemplated by the theorem is simply that of a prism; and in the case of a solid generated by a surface, the centre of gravity of the given surface is conceived to move along a curve line situated on a plane; and the moving surface is supposed to continue at right angles to the curve, which is evidently the extension which Leibnitz gave to the method of Guldinus.

But it is easy to see that the language in which Venturoli has expressed his theorem, is by no means limited to the simple cases which we have stated. If a straight line move parallel to itself, its centre of gravity, or middle point describing a straight line, the

surface described is simply a plane figure or rectangle, which is the only surface to which the theorem of Leibnitz can be applied, when the line described by the centre of gravity is straight. But it is obvious that the moving straight line may continue at right angles to the fixed straight line, and have, besides, an angular rotation about its centre of gravity; by which means we shall have produced a surface very different from a plane arc, and to such a surface the theorem just quoted will not apply, although it is comprehended in the terms of the proposition. On the whole, the statics of Venturoli contains much that deserves commendation, and very little that is justly liable to censure.

In his second book, which treats of motion, the author divides his subject into three sections. In the first section he considers the motion of a point, or single particle of matter, and discusses the principal elementary cases in a concise, but able and luminous manner. He gives the theory of equable and variable motions—of such as are uniformly accelerated or retarded—of vertical motions in free space and in resisting mediums;—then curvilineal motions; as, of bodies projected obliquely in free space or in air—of a point on a given curve line or surface on inclined planes—in a cycloid—in circular arcs—simple pendulum.

In his second section he treats of the motion of systems of invariable forms. Under this head he considers the motion of systems in general—moments of inertio—principal axes—motion about an immoveable axis—centre of percussion—of oscillation—initial movement of a rigid system—composition of rotatory motions.

In the third section is contained the doctrine of collision, or impact of bodies elastic and non-elastic, or imperfectly elastic—when the impact is central or eccentric, direct or oblique.

In this second part the author displays an extensive and accurate acquaintance with the theory of motion, which he exhibits in a natural, easy, and elementary manner, as well as the principal topics of discussion belonging to the science of dynamics. His brevity, however, on several articles, will render it necessary for the learner to consult other authors.

In the first chapter of the first section, we find the following account of variable motion and the measure of force.

"Variable motion is caused by a force which, acting continually on the moveable body, goes on at every instant increasing or diminishing its velocity. Such a force is called *accelerating* or *retarding*, and is measured at each instant by the ratio of that very small increment or decrement of velocity which it causes in the moveable body, to that very small time in which the increment is caused."

Now this is true, when the accelerative force is constant, but cannot be considered as perfectly correct when the force is variable. Our author evidently uses the language of the differential system of Leibnitz, and thus loses the great advantage that attends the genuine fluxions of Newton, or the theory of limiting ratios, which is equally correct, but not quite so easily demonstrated by

beginners. We have the true measure of the accelerative force by conceiving it to retain its quantity for any increment of time; and the augmentation of velocity which *would* thus be uniformly produced, divided by the increment of time, is the proper measure of the force. Or if we divide the *actual* increment of velocity by the increment of time, the quotient will approach continually to the measure of the accelerative force, as the element of time is taken smaller; and the limit of this quotient is precisely the measure of the force required.

But the learner has nothing to fear respecting the accuracy of the formula or equations, which the author immediately subjoins, for the purpose of exhibiting analytically the general relations of force, space, and time. In the second proposition of this first chapter, he denotes the accelerative force by φ, the velocity by u, the space by s, and the time by t; and the fluxions of s and t being denoted by ds and dt, he gives the fundamental equations for the motion of a single particle as follows:

$$u=\frac{ds}{dt}; \qquad \varphi=\frac{du}{dt}:$$

From these, by eliminating t, he has.....$\varphi ds=udu$,

and by eliminating u, he obtains.....$\varphi dt=d\frac{ds}{dt}$.

In the fifth chapter are given the general equations of the motion of a particle of matter when acted on by any forces. Here the brevity with which the subject is presented to the reader leads to obscurity; for in the general investigation, it is not indicated whether the co-ordinates may be oblique or rectangular; but in the second corollary to the general proposition we are informed, that "if the co-ordinates be at right angles to each other, then $udu=Pdx+Qdy+Rdz$;" in which it is implied, that the preceding general formula of this chapter might be used with oblique co-ordinates. This inference, however, will lead to error; for the three general equations of curvilineal motion given by the author in this place, are applicable only to rectangular co-ordinates.

In the seventh chapter we find a simple and beautiful investigation of the Ballistic curve, by means of differential equations. The author imitates the method of La Place in several parts of his investigation; as in first obtaining the general equations of the motion involving dt; afterwards in making dx constant instead of dt, by which means dt is eliminated, and the general equation is had in terms of the curve and co-ordinates; and, lastly, in his integration of the curve involving differential equations of the third degree.

(*To be continued.*)

ARTICLE III.

NEW QUESTIONS,

TO BE RESOLVED BY CORRESPONDENTS IN NO. II.

QUESTION I.—*By Mr. Charles Vyse.*

TEN pounds a quarter are allowed to five auditors, A, B, C, D, E, of a Fire Office. They are required to attend seven times in the quarter; and the absentees' shares are to be divided equally among such as attend. Now A and B never fail to attend, C and D are each absent twice, and E is absent only once: what is each auditor's share of the given sum?

N. B. This question having given rise to several disputes, is respectfully submitted to the judgment of skilful arithmeticians.

QUESTION II.—*By Philip W. Hanson, New-York.*

The product of two numbers is 10, and the product of their sum by the sum of their squares is 203. Quere, the numbers.

QUESTION III.—*By Mr. Dennis Leonard, New-York.*

To find two numbers such, that the sum of their squares multiplied into their difference may be equal to six times their product; and the difference of their fourth powers multiplied into the difference of their squares, may be equal to twenty-four times their product multiplied into their difference.

QUESTION IV.—*By Mr. James Ryan,* New-York.*

Given,

$$a^{2x} + a^{x} b^{y} + b^{2y} = m,$$

$$a^{2x} + 2a^{x} b^{y} + b^{2y} = m' - 2a^{x} - 2b^{y};$$

Required the values of x and y.

QUESTION V.—*By Mr. John Rochford, New-York.*

To find three numbers such, that the sum of the first and second, and also the difference of the first and third, shall be squares, the sum of whose roots shall be a square equal to the sum of the three required numbers.

QUESTION VI.—*By Diophantus.*

To find a rational right angled triangle such, that the hypothenuse diminished by each of the legs may be a perfect cube.

* This gentleman is the author of an ingenious and useful work lately published, entitled, "An Elementary Treatise on Algebra, practical and theoretical; adapted to the instruction of Youth in Schools and Colleges."

QUESTION VII.—*By the same.*

To find a rational right angled triangle such, that one leg may be a cube, the other a cube diminished by its side, and the hypothenuse a cube increased by its side.

N. B. These two questions are extracted from the Algebra of the celebrated mathematician Diophantus of Alexandria; they are the first and last questions of his sixth Book.

QUESTION VIII.—*By Patterson Wallace, New-York.*

Given the perpendicular of an equilateral triangle let fall from the vertical angle on the base, to construct the triangle.

QUESTION IX.—*By the same.*

Given the sum of the legs of a right angled triangle; and if the segments of the hypothenuse made by a perpendicular let fall upon it from the right angle be made the legs of another right angled triangle, the hypothenuse of this second triangle is also given: it is required to determine each of the triangles.

QUESTION X.—*By Mr. John Rochford, New-York.*

To find an arc of a given circle, the sum or difference of whose sine and versed sine is a maximum.

QUESTION XI.—*By Mr. James Ryan, New-York.*

Let $x^4-a^2x^2+a^2y^2=0$, be the equation of a curve, whose abscissa AP$=x$, and ordinate PM$=y$, on which is described the equilateral triangle PMQ: required the quadrature of the curve which is the locus of the point Q.

QUESTION XII.—*By a Member of the Mathematical Club, New-York.*

From a given cone to cut off a frustum such, that if the frustum be divided into two parts or ungulas by a plane touching the opposite bases, the difference of these ungulas may be a maximum.

QUESTION XIII.—*By Eboracensis.*

To find the greatest rectangle that can be inscribed in one of the ovals or nodi of the lemniscate, of which the equation is

$$a^2y^2=a^2x^2-x^4.$$

QUESTION XIV.—*By the same.*

To find the content of the solid formed by the revolution of the lemniscate about the axis of y, the equation of the curve being

$$a^2y^2=a^2x^2-x^4.$$

QUESTION XV.—*By Mr. B. M'Gowan, New-York.*

Given the day of the month, the perpendicular height of an object, and the transverse axis of the curve described on the hori-

zontal plane by the extremity of the shadow of the object, to determine the latitude of the place of observation.

QUESTION XVI.—*By a Member of the Mathematical Club, New-York.*

In plane sailing, the distance, difference of latitude, and departure, are the three sides of a right angled plane triangle, of which the angle contained between distance and difference of latitude is equal to the course; or, which amounts to the same thing,—as radius to the cosine of the course, so is distance to difference of latitude. Now it is required to determine whether this fundamental stating of plane sailing be strictly true, supposing the figure of the earth to be spherical, or to be an oblate spheroid of revolution, as it is very nearly, according to the sentiments of the most eminent mathematicians.

N. B. Teachers of Navigation are very much divided in opinion respecting the truth of the rules of plane sailing: on this account we have offered the problem to the consideration of men of science.

QUESTION XVII.—*By the same.*

What is the greatest difference that can occur between the distances of two places situated on the same parallel of latitude; one of those distances being reckoned on the parallel of latitude, and the other on the circumference of a great circle passing through the two places; supposing the earth to be spherical, and the diameter 7920 American miles?

QUESTION XVIII.—*By A. B. Quinby.*

If a common weighing scale with its contents weigh 60 pounds, and is suspended by three chains, each 24 inches in length, which unite in one point of suspension, and terminate in three points of the scale, at the equal distances of 12 inches from each other, the stress on each of the chains is required, supposing the chains to be equally inclined to the horizon.

QUESTION XIX.—*By Professor Adrain.*

It is required to investigate the nature of the curve described by a body projected obliquely along a given inclined plane, the resistance arising from friction being taken into consideration.

QUESTION XX.—**OR PRIZE QUESTION.**

By Mr. P. Fleming, Civil Engineer and Surveyor, New-York.

If at a point within a given circle, the three angles be observed which are subtended by three contiguous arcs of the circumference, of which the two extreme arcs are given in magnitude, but not in position; it is required to determine the magnitude of the intermediate arc, and the situation of the point where the observations are taken.

THE

MATHEMATICAL DIARY,

N° II.

BEING THE PRIZE NUMBER OF NATHANIEL BOWDITCH, LL.D. OF BOSTON.

ARTICLE V.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE III. NO. I.

QUESTION I.—*By Mr. Charles Vyse.*

TEN pounds a quarter are allowed to five auditors, A, B, C, D, E, of a Fire Office. They are required to attend seven times in the quarter, and the absentees' shares are to be divided equally among such as attend. Now A and B never fail to attend, C and D are each absent twice, and E is absent only once: what is each auditor's share of the given sum?

SOLUTION.—*By Mr. James Phillips, Harlem.*

It is evident, from the nature of the question, that it admits of as many answers as there are different ways in which C, D and E can absent themselves. Now these men can only be placed in six positions; consequently, if C and D, at each time, receive equal shares, the number of answers will be limited to six. Yet they may receive different positions, because they can be so placed that C's share will become D's and D's become C's, a mutation which produces no alteration of the several sums distributed, but merely a change of recipients; and thus the whole number of solutions appears to be eight.

First, if three be absent the first night of meeting, and two on the following night, then A will receive $\frac{110}{42}l.$; B $\frac{110}{42}l.$; C $\frac{60}{42}l.$; D $\frac{60}{42}l.$; and E $\frac{90}{42}l.$

Mr. Phillips then goes on to show all the eight different ways of attendance, and calculates the eight sets of answers. Several other gentlemen also point out the varieties which occur in the answers.

QUESTION II.—*By Philip W. Hanson, New-York.*

The product of two numbers is 10, and the product of their sum by the sum of their squares is 203. Quere, the numbers.

FIRST SOLUTION.—*By Mr. B. Mc.Gowan, New-York.*

Let $x=\frac{1}{2}$ the sum and $y=\frac{1}{2}$ the difference of the required numbers, then, by the conditions of the question, we have $x^2-y^2=10$ and $(2x^2+2y^2).2x=203$, or $4x^3+4xy^2=203$. Now multiply the first equation by $4x$ and it becomes $4x^3-4xy^2=40x$, which equation added to the 2d, the sum is $8x^3=40x+203$, or $8x^3-40x=203$. From this cubic we obtain $x=3.5$, and consequently by the first equation $y=1.5$, and 5 and 2 are the numbers required.

SECOND SOLUTION.—*By Mr. Dennis Leonard, N. York.*

Let s the sum of the required numbers and $p=$ their product; then $s^2-2p=$ the sum of their squares, and by the question

$$s(s^2-2p)=s^3-20s=203$$

whence $s=7$, now having the sum 7 and product 10, the numbers required are easily found to be 5 and 2.

QUESTION III.—*By Dennis Leonard, New-York.*

To find two numbers such, that the sum of their squares multiplied into their difference may be equal to six times their product; and the difference of their fourth powers multiplied into the difference of their squares, may be equal to twenty-four times their product multiplied into their difference.

SOLUTION.—*By Mr. Joseph C. Strode, Chester Co., Pa.*

Let $x+y$ and $x-y$ be the two numbers sought; then the conditions of the question will be expressed by the equations,

$$2y(2x^2+2y^2)=6(x^2-y^2),$$
$$4xy(8x^3y+8xy^3)=48y(x^2-y^2).$$

Divide the latter of these equations by the former, and we have $8x^2y=8y$, whence $x^2=1$, and $x=1$. Substitute this value of x in the first equation and it becomes by reduction

$$y^3+\tfrac{3}{2}y^2+y=\tfrac{3}{2},$$

whence $y=.636$, and $x+y=1.636$, $x-y=1-.636=.364$, which are the numbers sought.

QUESTION IV.—*By James Ryan, New-York.*

Given

$$a^{2x}+a^xb^y+b^{2y}=m,$$

$$a^{2x}+2a^xb^y+b^{2y}=m'-2a^x-2b^y;$$

required the values of x and y.

SOLUTION.—*By Mr. Daniel Shanley, Charleston, S. C.*

From the 2d equation, by transposition, completing the square, &c.

$$a^x+b^y=\pm\sqrt{m'+1}-1=p:$$

squaring this equation, and taking the first from it, the resulting equation, is

$$a^xb^y=p^2-m$$

Now subtracting three times this last equation from the first, and extracting the square root, $a^x - b^y = \pm\sqrt{4m - 3p^2} = q$.

By addition and subtraction, $a^x = \frac{1}{2}p + \frac{1}{2}q$, $b^y = \frac{1}{2}p - \frac{1}{2}q$.
Hence, taking logarithms, &c.

$$x = \frac{\log.(\frac{1}{2}p + \frac{1}{2}q)}{\log.\ a}, \qquad y = \frac{\log.(\frac{1}{2}p - \frac{1}{2}q)}{\log.\ b}.$$

QUESTION V.—*By Mr. John Rochford, New-York.*

To find three numbers such, that the sum of the first and second, and also the difference of the first and third, shall be squares, the sum of whose roots shall be a square equal to the sum of the three required numbers.

—

FIRST SOLUTION.—*By Mr. Farrand N. Benedict, of Montezuma, State of New-York.*

Let the numbers sought be $a^2x^2 - x^2$, x^2, and $(b^2 + a^2 - 1)\,x^2$. By an assumption of this kind, the two first conditions of the question are obviously satisfied; the sum of the two first being a^2x^2, and the difference of the first and third b^2x^2. Again, per question, $(a+b)x = (2a^2 + b^2 - 1)x^2$, therefore $x = \frac{a+b}{2a^2+b^2-1}$.

It now remains to make $(a+b)x = \frac{(a+b)^2}{2a^2+b^2-1} =$ a square.

To effect this we have only to make $2a^2 + b^2 - 1$ a square.
Assume $2a^2 + b^2 - 1 = b^2 + m^2 - 2bm$, which by reduction gives $b = \frac{m^2 - 2a^2 + 1}{2m}$, where a may be assumed any number greater than unity.

—

SECOND SOLUTION.—*By Mr. John Capp of Harrisburg.*

Let $8x^2$, x^2, $7x^2$, denote the numbers, then the sum of the first and second $= 9x^2 = (3x)^2$, and the difference of the first and third, $= 8x^2 - 7x^2 = x^2$, and the roots of these $= 4x = 16x^2$, which is a square, whence $x = \frac{1}{4}$, hence $\frac{8}{16}$, $\frac{1}{16}$, $\frac{7}{16}$, are the numbers required.

QUESTION VII.—*By Diophantus.*

To find a rational right angled triangle such that the hypothenuse diminished by each of the legs may be a perfect cube.

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FIRST SOLUTION.—*By Mr. Farrell Ward, New-York.*

Let $m^2 + n^2$, $m^2 - n^2$, and $2mn$, represent the hypothenuse, base, and perpendicular of the right angled triangle; then per question, $(m^2 + n^2) - (m^2 - n^2) = 2n^2 =$ a cube $= n^3$, whence $n = 2$; again, $m^2 + n^2 - 2mn =$ a cube $= (m-n)^3$, and dividing by $(m-n)^2$, we have $m - n = 1$, whence $m = 3$, and by substitution the numbers are found, 5, 12 13, which are sides of the right angled triangle.

SECOND SOLUTION.—*By Mr. Roche, Philadelphia.*

Put s^2+b^2, s^2-b^2 and $2sb$ for the sides of the right angled triangle required; then the first less the second is $2b^2=$ a cube per question$=r^3$, hence $b=\frac{r}{2}\sqrt{2r}$. Again, the first less the third is $s^2-2sb+b^2=$ a cube, $=v^3$, whence $s-b=\sqrt{v^3}=v\sqrt{v}$; it is evident, therefore, that we may assume for $2r$ and v any two rational squares. Thus, if $2r=4$, $v=1$ we have $b=2$ and $s=3$, whence 13, 12 and 2 are the sides, and are the least whole numbers that will answer the question.

QUESTION VII.—*By the same.*

To find a rational right angled triangle such, that one leg may be a cube, the other a cube diminished by its side, and the hypothenuse a cube increased by its side.

SOLUTION.—*By Mr. B. McGowan, New-York.*

Putting $x^3=$ the base of the right angled triangle, x^3+x and x^3-x will represent the hypothenuse and perpendicular, therefore

$$x^6=(x^3+x)^2-(x^3-x)^2,$$

hence $4x^4=x^6$, or $x^2=4$, therefore $x=2$, consequently 6, 8, and 10 are the three sides required.

QUESTION VIII.—*By Patterson Wallace, New-York.*

Given, the perpendicular of an equilateral triangle let fall from the vertical angle on the base, to construct the triangle.

SOLUTION.—*By Master Walter Patterson, Classical School, Hudson-street, New-York.*

At any given point, in an indefinite straight line erect a straight line equal to the given perpendicular, and at the upper extremity of this straight line make on each side of the line an angle equal to half the angle of an equilateral triangle, and producing the straight lines containing these angles until they meet the indefinite straight line, the required equilateral triangle is constructed.

QUESTION IX.—*By the same.*

Given, the sum of the legs of a right angled triangle, and if the segments of the hypothenuse made by a perpendicular let fall upon it from the right angle be made the legs of another right angled triangle, the hypothenuse of this second triangle is also given: it is required each of the triangles.

SOLUTION—*By Mr. Rochford, New-York.*

Admit $2a=$ the sum of the legs, and $2x$ their difference, then $a+x$, and $a-x$ are the legs, the hypothenuse$=(2a^2+2x^2)^{\frac{1}{2}}$, and the segments of do. made by the perpendicular, are

$$\frac{(a+x)^2}{(2a^2+1x^2)^{\frac{1}{2}}} \text{ and } \frac{(a-x^2}{(2a^2+2x^2)^{\frac{1}{2}}}$$

and therefore, by the question, putting the given hypothenuse equal to b, we have

$$\frac{(a+x)^4}{2a^2+2x^2} + \frac{(a-x)^4}{2a^2+2x^2} = b^2,$$

whence $x^4+(6a^2-6b)x^2=a^2(b^2-a^2)$,
and therefore by completing the square, &c. we have

$$x=\left\{\frac{b^2}{2} - 3a^2+(8a^4-2a^2b^2+\frac{b^4}{4})^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

and consequently the sides of the triangle are found.

QUESTION X.—*By Mr. John Rochford, New-York.*

To find an arc of a given circle, the sum or difference of whose sine and versed sine is a maximum.

SOLUTION.—*By Mr. P. Fleming, New-York.*

Put the angle sought, A, and by the question,

$$\text{sine A}+(1-\text{cos. A})=\text{maximum.}$$

By differentiating, we have cos. A+sine A=0: now the angle, of which the sine is equal to the cosine, is 45°; hence

$$90°+45°=135°=\text{A.}$$

Also by the question, sine A—1+cos. A=max.
differentiating, cos. A—sine A=0,
or cos. A=sine A, and A=45°.

QUESTION XI.—*By Mr. James Ryan, New-York.*

Let $x^4-a^2x^2+a^2y^2=0$, be the equation of a curve, whose abscissa AP=x, and ordinate PM=y, on which is described the equilateral triangle PMQ: required the quadrature of the curve which is the locus of the point Q.

SOLUTION.—*By Mr. Charles Wilder, Baltimore.*

It is plain that the co-ordinates y' and x' of the new curve, are $y'=\frac{y}{2}$, and $x'=x\pm y\frac{\sqrt{3}}{2}$; hence

$$\dot{x}'=\dot{x}\pm\dot{y}\frac{\sqrt{3}}{2};$$

and therefore, by multiplication we have

$$y'\dot{x}'=\frac{y\dot{x}}{2}+\frac{y\dot{y}\sqrt{3}}{4}.$$

Now, if A = the area of lemniscate, that is, $A=\int y\dot{x}$, we have by substitution, and taking the fluent,

$$\int y'\dot{x}'=\frac{A}{2}+\frac{y^2\sqrt{3}}{8},$$

which is the area required.

QUESTION XII.—*By a Member of the Mathematical Club, New-York.*

From a given cone to cut off a frustum such, that if the frustum be divided into two parts or ungulas by a plane touching the opposite bases, the difference of these ungulas may be a maximum.

SOLUTION.—This question is different from that which was intended, the word "*difference*" having been put instead of the word "*less.*"

QUESTION XIII.—*By Eboracensis.*

To find the greatest rectangle that can be inscribed in one of the ovals or nodi of the lemniscate, of which the equation is

$$a^2y^2=a^2x^2-x^4.$$

FIRST SOLUTION.—*By Professor Strong, Hamilton College.*

The conditions of the question require, that

$$2y\left\{\sqrt{a^2+a\sqrt{a^2-4y^2}}-\sqrt{a^2-a\sqrt{a^2-4y^2}}\right\}=\text{a max.}$$

This being transformed and reduced, we have

$$ay^2-y^3=\text{max.},$$

which, by taking the fluxion, &c. gives $y=\frac{1}{3}a$.

[N. B. Mr. Wilder's solution is very nearly the same in method and operations with that of Professor Strong, and is equally ingenious.—EDITOR.]

SECOND SOLUTION.—*By Dr. Nathaniel Bowditch, Boston.*

Assume $x=a$. sine z, then $y=a$. sine z . cos. $z=\frac{a}{2}$. sine $2z$: and y' being the opposite and equal ordinate with y, and z' the corresponding angle, we have $y'=\frac{a}{2}$. sine $2z'$, therefore sine $2z'=$ sine $2z$, and consequently $z'=90°-z$. Thus $x'=a$ sine $z'=a$ cos. z, and $x'-x=a$ (cos. z—sine z), the abscissa x' corresponding to z'. Now multiply by $2y=a$ sine $2z$, the area of the rectangle is

$$a^2 \text{ . sine } 2z \text{ (cos. } z-\text{sine } z),$$

whose differential put $=0$, and divided by $a^2 . dz$, gives

$$0=2 \text{ cos. } 2z \text{ (cos. } z+\text{ sine } z)-\text{sine } 2z \text{ (sine } z+\text{ cos. } z);$$

and as cos. $2z=$cos.$^2 z-$sine$^2 z=$(cos. $z+$ sine z) . (cos. $z-$sine z), the preceding expression becomes

$$0=(\text{cos. } z+\text{ sine } z) . \{2 \text{ (cos. } z-\text{sine } z)^2-\text{sine } 2z\}.$$

But z being $< 90°$, cos. $z+$sine z is positive, and cannot become $=0$; therefore

$$0=z \text{ (cos. } z-\text{sine } z)^2-\text{sine } 2z,$$

$$0=2 \text{ (cos.}^2 4+\text{sine}^2 z-2 \text{ cos. } z \text{ sine } z)-\text{sine } 2z,$$

or

$$0=(2-2 \text{ sine } 2z)-\text{sine } 2z,$$

hence sine $2z=\frac{2}{3}$; therefore $y=\frac{1}{2}a$. sine $2z=\frac{a}{3}$.

QUESTION XIV.—*By the same.*

To find the content of the solid formed by the revolution of the lemniscate about the axis of y, the equation of the curve being

$$a^2y^2=a^2x^2-x^4.$$

SOLUTION.—*By Mr. Wilder, Baltimore.*

The equation solved, gives the two values of x^2, viz.

$$x^2=\frac{a^2}{2}+\frac{a}{2}\sqrt{a^2-4y^2}, \text{ and } x^2=\frac{a^2}{2}-\frac{a}{2}\sqrt{a^2-4y^2},$$

the difference of which, multiplied by $\pi=3.14159$, &c., gives the area of the generating plane $=\pi a\sqrt{a^2-4y^2}$; consequently the fluxion of the required solid is

$$\pi a\cdot\dot{y}\cdot\sqrt{a^2-4\dot{y}^2},$$

whose fluent is $\frac{\pi a}{2}\times$ the circular segment to radius a and sine $2y$, no correction being necessary; and the whole solid described by an oval $=\frac{\pi^2a^3}{4}$.

QUESTION XV.—*By Mr. B. M'Gowan, New-York.*

Given the day of the month, the perpendicular height of an object, and the transverse axis of the curve described on the horizontal plane by the extremity of the shadow of the object, to determine the latitude of the place of observation.

FIRST SOLUTION.—*By Professor Strong, Hamilton College.*

Let L = the latitude sought, D = the declination, h = the given height, t = the given transverse; then by the conditions of the problem, we have

$$h\{\text{tan. (L+D)}-\text{tan. (L}-\text{D)}\}=t;$$

whence $\quad \text{cos. L}=\pm\frac{1}{t}\sqrt{h\,\text{sine 2D}+t(1-\text{cos. 2D})},$

which shows the cosine of the latitude required.

SECOND SOLUTION.—*By Mr. John Rochford, New-York.*

By Art. 467 of *Simpson's Fluxions*, we reduce the problem to the following in plane geometry:

Given the base of plane triangle = the given transverse, the perpendicular from the vertical angle on the base = the given height of the object, and the vertical angle = the supplement of the sun's declination on the given day to construct the triangle.

The construction and calculation of this triangle are given in Prob. V., Appendix to Simpson's Algebra, which it is not necessary to repeat.

[In the preceding solution, the author confines himself to the case in which the sun, in his diurnal motion, does not cut the horizon, and of course the curve described by the extremity of the shadow is an ellipse. When the sun's parallel of declination is cut by the horizon, the curve is an hyperbola, the vertical angle of the plane triangle to be constructed is double the sun's declination, and the difference of the angles at the base is double the required latitude If the transverse axis of the curve be given equal to infinity, the curve will be a parabola, and the latitude of the place will be equal to the complement of the sun's declination.—Ed.]

QUESTION XVI.—*By a Member of the Mathematical Club, New-York.*

In plane sailing, the distance, difference of latitude, and departure are the three sides of a plane triangle, of which the angle contained between distance and difference of latitude is equal to the course; or, which amounts to the same thing,—as radius to the cosine of the course, so is distance to difference of latitude. Now it is required to determine, whether this fundamental stating of plane sailing be strictly true, or to be an oblate spheroid of revolution, as it is very nearly, according to the sentiments of the most eminent mathematicians.

[This important question has been discussed by many learned mathematicians in Europe, who have demonstrated in the most satisfactory manner the perfect accuracy of the statings in plane sailing, when the earth is supposed to be a sphere; but when the earth is supposed to be a spheroid of revolution, their reasoning, on which the statings depend, appears to me to be objectionable.

To show the nature of my objection, let C= the course, and D = the distance on a loxodromic line, reckoning from any given point on the surface of the earth, which is supposed to be a spheroid of revolution; also let E= the arc of the elliptic meridian, intercepted between the parallels of latitude passing through the fixed and variable extremities of the distance D, and let P= the departure, corresponding to D.

Then, it is admitted that $\dot{E} = \dot{D}.\cos.C$, $\dot{P} = \dot{D}.\sin.C$, and therefore by integrating we have

$$E = D.\cos.C, \quad P = D.\sin.C,$$

no arbitrary constants being necessary, because D, E, and P begin together.

Now these equations expressing the values of E and P show that the distance D, the arc of the meridian E, and the departure P are truly represented by the three sides of a right angled plane triangle, having the angle C opposite to the side P, the hypothenuse being D. In this triangle we have the following statings:

Rad : Sin. Course :: dist. D : departure P.
Rad : Cos. Course :: dist. D : meridional arc E.

Now in the case of a sphere, the meridional arc E is the measure of the difference of latitude; but in the spheroid this meridional arc E is not the measure of the difference of latitude, nor proportion to it; the difference of latitude being the measure of the angle

contained between two verticals in the same meridian, passing through the extremity of the arc E. Since therefore the difference of latitude is measured by a quantity greater or less than E, therefore the fundamental stating of plane sailing is erroneous, when the earth is a spheroid of revolution. EDITOR.]

QUESTION XVII.—*By the same.*

What is the greatest difference that can occur between the distances of two places situated on the same parallel of latitude; one of these distances being reckoned on the parallel of latitude, and the other on the circumference of a great circle passing through the two places; supposing the earth to be spherical, and the diame- 7920 American miles?

SOLUTION.—*By Eboracensis.*

The difference of longitude being evidently 180°, let $z=$ the polar distance, $\pi=3.14159$ &c. and by the question,

$$\frac{\pi}{2}\text{Sin. } z - z = \text{max.}$$

Therefore,

$$\frac{\pi}{2} dz \,.\text{Cos. } z - dz = 0,$$

whence dividing by dz we have $\cos. z=\frac{2}{\pi}$, or $\sec. z=\frac{1}{2}\pi$, that is cosec. latitude $=\frac{1}{2}\pi=1.5707963$, and therefore by the table of natural secants the latitude is had by inspection, $=39°\ 32'\ 24''.8$.

Corollary 1. The secant of the polar distance is equal to a quadrant of the meridian.

Cor. 2. This problem is resolved very simply by geometrical differentials of the polar distance and its ordinate or sine giving the ratio as $1 : \frac{1}{2}\pi$ :: rad. : sec. pol. dis. or cosec. of lat.

Cor. 3. This geometrical method also shews with equal facility, that we have precisely the same stating and same latitude in an oblate spheroid of revolution.

QUESTION XVIII.—*By Mr. A. B. Quinby, New-York.*

If a common weighing scale with its contents weigh 60 pounds, and is suspended by three chains, each 24 inches in length, which unite in one point of suspension, and terminate in three points of the scale, at the equal distance of 12 inches from each other: the stress on each of the chains is required, supposing the chains to be equally inclined to the horizon.

SOLUTION.—*By Mr. P. Fleming, New-York.*

Let A= the angle which each chain makes with the vertical, and P= power or force with which each chain is stretched, then we have by statics,

$$P\cos.A + P\cos.A + P\cos.A = 60\text{lb.}$$

that is $3P\cos.A = 60$, hence $P = \frac{20}{\cos.A} = 20.89\text{lb.}$

QUESTION XIX.—*By Professor Adrain.*

It is required to investigate the nature of the curve described by a body projected obliquely along a given inclined plane, the resistance arising from friction being taken into consideration.

FIRST SOLUTION.—*By Professor Strong.*

I shall assume the formulæ given by La Place, Mec. Cel. vol. 1 page 26, (which obviously apply here, supposing the gravity down the given plane to act with an invariable intensity, and in directions perpendicular to a horizontal line drawn on the plane.)

$$\frac{F}{2g'} = \mp\frac{ds\,d^2z}{2(d^2z)^2} \quad , \quad 2g'dt^2 = \mp\frac{ds\,dx}{d^2z}.$$

where $2g' = 32\frac{1}{6} \times s'$, s = sine of the plane's inclination, rad. 1, s = the portion of the curve described from the commencement of the motion, x and z being the rectangular co-ordinates corresponding, having their origin at the point of projection, x being reckoned on a horizontal line passing through that point, and z on a line at right angles to x drawn on the plane through the same point; F denotes the friction, the sign — to be used if the body be projected up the plane, and the sign + if it be projected down it. I shall suppose F to be constant on a plane of given inclination, which it appears to be or nearly so, by the experiments of *Vince* and *Coulomb*.

Put $\frac{F}{2g'} = m$, and $\frac{dz}{dx} = p$, (dx being constant) and our formulæ become by substitution

$$\frac{m\,dp}{\sqrt{1+p^2}} = \mp\frac{ddp}{dp} \quad , \text{ and } 2g\,dt^2 = \mp dp\,dx.$$

The first integrated gives

$$-dp(p+\sqrt{1+p^2})^m = \frac{dx}{c} \quad , \text{ and } dpp(p+\sqrt{1+p^2})^m = \frac{dz}{c},$$

$$dp(+p\sqrt{1+p^2})^m = \frac{dx}{c'} \quad , \text{ and } dpp(\sqrt{1+p^2})^m = \frac{dz}{c'} \quad ,$$

according as the body is projected up or down the plane.

$$\left(c = \frac{V}{2g'(1+p').(p'+\sqrt{1+p'^2})}, \text{ and } c = \frac{V}{2g'(1+p'^2).(\sqrt{1p'^2}-p')^m} \right.$$

$\left.\right.$ V being the velocity of projection, and p' the tangent of the angle of projection with the axis of x, rad. 1. $\left.\right)$

By integrating and correction once more, we have

$$\frac{2x}{c}=\frac{(p'+\sqrt{1+p'^2})^{m+1}-(p+\sqrt{1+p^2})^{m+1}}{m+1}+\frac{(p'+\sqrt{1+p'^2})^{m-1}-(p+\sqrt{1+p^2})^{m-1}}{m-1},$$

and

$$\frac{4z}{c}=\frac{(p'+\sqrt{1+p'^2})^{m+2}-(p+\sqrt{1+p^2})^{m+2}}{m-2}+\frac{(p+\sqrt{1+p^2})^{m+2}-(p'+\sqrt{1+p^2})^{m+2}}{m-2}$$

which equations apply when the body is projected up the plane: by assuming for p any value less than p' in these equations we have x and z, p being positive for the ascending branch of the curve, and negative for the descending branch; the vertex is found by making $p=0$.

If the body be projected down the plane, then

$$\frac{2x}{c'}=\frac{(\sqrt{1+p'^2}-p')^{m+1}-(\sqrt{1+p^2}-p)^{m+1}}{m+1}+\frac{(\sqrt{1+p'^2}-p')^{m-1}-(\sqrt{1+p^2}-p)^{m-1}}{m-1},$$

and

$$\frac{4z}{c'}=\frac{(\sqrt{1+p^2}-p)^{m+2}-(\sqrt{1+p'^2}-p')^{m+2}}{m+2}+\frac{(\sqrt{1+p'^2}-p')^{m-2}-(\sqrt{1+p^2}-p)^{m-2}}{m-2};$$

p in this case being greater than p'.

If $m=1$, then in the first case we have

$$\frac{2x}{c}=\frac{(p'+\sqrt{1+p'^2})^2-(p+\sqrt{1+p^2})^2}{2}+h.\,l.\frac{p'+\sqrt{1+p'^2}}{p+\sqrt{1+p'^2}}$$

The same expressions answer to the second case, except that the signs of the simple powers of p' and p are to be changed.

If $m=2$, then in the first case,

$$\frac{4z}{c}=\frac{(p'+\sqrt{1+p'^2})^4-(p+\sqrt{1+p^2})^4}{4}+h.\,l.\frac{(p+\sqrt{1+p^2})}{p'+\sqrt{1+p'^2}};$$

and in the second case,

$$\frac{4z}{c'}=\frac{(\sqrt{1+p^2}-p)^4-(\sqrt{1+p'^2}-p')^4}{4}+h.\,l.\frac{(\sqrt{1+p'^2}-p')}{\sqrt{1+p^2}-p}$$

I have set down these expressions, because they make an exception to the general integrations previously given.

Again, the velocity,

$$v = V \cdot \sqrt{\frac{1+p^2}{1+p'^2}} \times \left(\frac{\sqrt{1+p^2} \pm p}{\sqrt{1+p'^2} \pm p'} \right)^{\frac{m}{2}} ;$$

V being the velocity of projection, and v the velocity at any other point, the sign $+$ is to be used when the body ascends, and the sign $-$ when it descends.

The time,

$$t = \frac{(p' + \sqrt{1+p'^2})^{\frac{m}{2}+1} - (p + \sqrt{1+p^2})^{\frac{m}{2}+1}}{2(\frac{m}{2}+1) \times \sqrt{\frac{2g'}{c}}} +$$

$$\frac{(p' + \sqrt{1+p'^2})^{\frac{m}{2}-1} - (p + \sqrt{1+p^2})^{\frac{m}{2}-1}}{2(\frac{m}{2}-1) \times \sqrt{\frac{2g'}{c}}}$$

in the first case, p being taken with $+$ in the ascending branch, and negative in the descending branch of the curve.

When $m=2$, the value of t is determined by the equation

$$t = \frac{(p' + \sqrt{1+p'^2})^2 - (p + \sqrt{1+p^2})^2}{4\sqrt{\frac{2g}{c}}} + h.\ l.\ \frac{(p' + \sqrt{1+p'^2})}{p + \sqrt{1+p^2}}$$

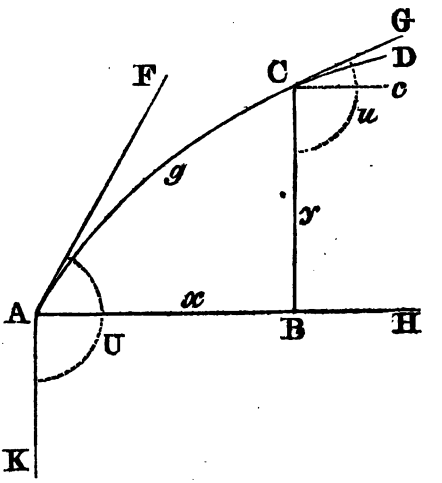
In the second case we have the same expressions, with the exceptions that the signs of p' and p are to be changed, and c' to be substituted for c. Should the friction vary as any power of the velocity, then the solution is to be effected after the manner of La Place in the place cited above.

SECOND SOLUTION.—*By Nathaniel Bowditch, LL.D. Boston.*

Let ACD be the proposed curve described on the given plane, which we shall take for the plane of the figure. A the point of projection in the direction AF a tangent to the curve at A. Through A draw the horizontal line ABH, and from any point C of the curve let fall upon this line the perpendicular ordinate CB, and let CG be the tangent to the curve at C. Through A draw downwards AK, parallel to CB.

Put AB$=x$, BC$=y$, AC$=s$, angle KAF$=$U, angle BCG$=a$, V$=$ the velocity of the body at A in the direction of AF, and $v=$ the velocity at C in the direction of the tangent CG. Moreover let $g=$ the space fallen through in a second by a body falling freely by the force of gravity, and $e=$ the angle of inclination of the proposed plane to the horizon, f the proportion of the

friction to the pressure, which is generally about $\frac{1}{3}$. Then the accelerating force of gravity in the plane is g sin. e, which for brevity I shall put $=b$; it acts in the direction of CB, and the resist-



ance equal $f g$ cos. e acts in the direction GC, this may be put $=a$, and if we put $m=\frac{a}{b}=$cot. e, this last force will be mb, and if we reduce it to the directions of the axis of x and y it will be $-mb\frac{dx}{ds}$, $-mb\frac{dy}{ds}$, therefore the accelerating force tending to increase x and y, are respectively $-mb.\frac{dx}{ds}$, $-b-mb\frac{dy}{ds}$, which by the usual rules of Dynamics are respectively equal to $d.\frac{dx}{dt}$, $d\frac{dy}{dt}$, t representing, as usual, the time. Hence we have the following fundamental equations to determine the nature of the curve.

$$d.\frac{dx}{dt}+mb.\frac{dx}{ds}dt=0 \qquad (1)$$

$$d.\frac{dy}{dt}+(b+mb.\frac{dy}{ds})dt=0 \qquad (2)$$

Multiply the first by $2\frac{dx}{dt}$, the second by $2\frac{dy}{dt}$, integrate and putting $dx^2+dy^2=ds^2$, and $\frac{ds}{dt}=v$ we get

$$v^2+2by+2bmx=V^2 \qquad (3)$$

V being taken for the constant quantity introduced by integration, observing that at the point A, where $v=V$, $y=0$, $s=0$.

As it would be complicated and frequently impossible to express x, y, s in terms of t, I shall proceed to express them all in terms of u. For this purpose suppose dx constant, and put $dy=pdx$, $dp=qdx$. Now if we develope the term $d.\frac{dx}{dt}$ of the equation (1) we easily obtain $mb=\frac{ds\,ddt}{dt^3}$, substituting this in equation (2) developed in like manner, and rejecting the terms which destroy each other, we get by reduction

$$ddy+bdt^2=0,$$

whose differential is $d^3y+2bdtddt=0$;

but we have just found $ddt=\frac{mbdt^3}{ds}$ hence

$$d^3y+\frac{2mbdt^4}{ds}=0,$$

substituting $bdt^2=-ddy$ this becomes

$$d^3y+\frac{2m(ddy)^2}{ds}=0$$

whence $$2m=-\frac{ds\,dy}{(ddy)^2}$$

This is reduced to terms of p, y, by putting as above $dy=pdx$, $ddy=dp.dx=qdx^2$, $d^3y=dqdx^2$, and $ds=\sqrt{dx^2+dy^2}=dx\sqrt{1+p^2}$ which being substituted, gives $2m=-\frac{\sqrt{1+p^2}.dq}{q^2dx}$;

and as $qdx=dp$, it becomes

$$2m=-\frac{\sqrt{1+p^2}.dq}{qdp}, \text{ or } \frac{dq}{q}=-\frac{-2mdp}{\sqrt{1+p^2}},$$

whose integral is $q=-\frac{1}{C}.\left\{p+\sqrt{1+p^2}\right\}^{-2m}$.

This may be put under a more simple form, by observing that $p=\frac{dy}{dx}=$ tan. GCc $=$ tan. $(u-90)=-$ cot. u. and $\sqrt{1+p^2}=\sqrt{1+\cot.^2u}=$ cosec. u, hence $p+\sqrt{1+p^2}=$ cosec. $u-$ cot. u, $=\frac{1-\cos.u}{\sin.u}=\frac{2.\sin.^2\frac{1}{2}u}{2\sin.\frac{1}{2}u\cos.u}=$ tan. $\frac{1}{2}u$, consequently $q=-\frac{1}{C}$ tan. $\frac{1}{2}u^{-2m}$.

To determine the constant C we shall observe that as above $q=\frac{ddy}{dx^2}=-\frac{bdt^2}{dx^2}$; but $\frac{dx}{dt}$ represents the velocity in the direction parallel to AH, and at the point A this is evidently equal to V sin. U, consequently at this point we shall have $q=-\frac{b}{V^2.\sin.^2U}$; putting this equal to $-\frac{2}{C}$. tan. $\frac{1}{2}u^{-2m}$, we get

$$C=\frac{V^2.\ \sin.\ U}{b.\ \tan.\ \frac{1}{2}u^{2m}},$$

consequently the original expression of q becomes

$$q=-\frac{1}{C}.(p+\sqrt{1+p^2})^{-2m}=-\frac{1}{C}\tan.\tfrac{1}{2}u^{-2m}=-\left(\frac{b.\ \tan.\ \frac{1}{2}U}{V^2.\ \sin.^2U}\right).\ \tan.\ \tfrac{1}{2}u^{-2m};$$

still using for brevity the constant C, and putting moreover $\tan.\ \frac{1}{2}u=x$, $\tan.\ \frac{1}{2}V=Z$, we shall have $\tan.\ u=\frac{2\tan.\ \frac{1}{2}u}{1-\tan.^2\frac{1}{2}u}=\frac{2x}{1-x^2}$, hence $p=-\cot.u=\frac{x^2-1}{x^2}=\frac{1}{2}x-\frac{1}{2}x^{-1}$; also $\sqrt{1+p^2}=\frac{1}{2}x+\frac{1}{2}x-1$, hence $dp=\frac{1}{2}dx(1+x^{-2})$ and $q=-\frac{1}{C}.x^{-2m}$, consequently we have

$$s=\int\frac{dp}{q}=\tfrac{1}{2}C\int dx(-x^{2m}-x^{2m-2})$$

$$=\tfrac{1}{2}C\left\{\frac{Z^{2m+1}-x^{2m+1}}{2m+1}+\frac{Z^{2m-1}-x^{m-1}}{2m-1}\right\},$$

the integral being taken so that $s=o$, when $x=Z$.

$$y=\int pdx=\int\frac{pdp}{q}=\int\tfrac{1}{4}\frac{xdx(1-x^{-4})}{2}$$

$$=\tfrac{1}{4}C\int dx(-x^{2m+1}+x^{2m-3}),\ \text{that is}$$

$$y=\left\{\frac{Z^{2m+2}-x^{2m+2}}{2m+2}-\frac{Z^{2m-2}-x^{2m-2}}{2m-2}\right\},$$

the integral being taken so that $y=o$, when $x=Z$.

$$s=\int dx\sqrt{1+p^2}=\int\frac{dp}{q}\sqrt{1+p^2}=-\tfrac{1}{4}C\int xdx(1+x^{-2})^2.x^{2m}$$

that is $s=\frac{1}{4}C\int dx(-x^{2m+1}-2x^{2m-1}-x^{2m-3})$

$$=\tfrac{1}{4}C\left\{Z^{2m+2}+2.\frac{Z^{2m}-x^{2m}}{2m}+\frac{Z^{2m-2}-x^{2m-2}}{2m-2}\right\}$$

which value of s also vanishes when $x=Z$.

The values of y, s being found, give $2by+2mbs=b(2y+2ms)=$

$$\tfrac{1}{4}Cb\left\{(Z^{2m+2}-x^{2m+2})+2(Z^m+x^m)+(Z^{2m-2}+x^{2m2})\right\}\ \text{or by reduction}$$

$$2by+mbs=\tfrac{1}{4}Cb\left\{Z^{2m-2}(Z^2+1)^2-x^{2m-2}(x^2+1)^2\right\}$$

But since $\tan.\ \frac{1}{2}U=Z$, $\sin.U=\frac{2Z}{1+Z^2}$, the above value of $C=\frac{V^2.\ \sin.\ U^2}{b\ \tan.\ \frac{1}{2}U^{2m}}$ gives $\frac{1}{4}Cb=\frac{V^2}{(1+Z^2).Z^{2m-2}}$, and consequently

$$2by+2mbs=V^2-v^2.\frac{z^{2m-2}.(z^2+1)^2}{Z^{2m-2}.(Z^2+1^2}$$

substituting this in the equation (3) and reducing, we get

$$v=V.\frac{z^{m-1}.(z^2+1)}{Z^{m-1}.(Z^2+1)}$$

Now substituting the value of C in x, y, z, collecting their values and that of v together, we get

$$v=V.\frac{z^{m-1}.(z^2+1)}{Z^{m-1}.(Z^2+1)} \qquad (4)$$

$$x=\frac{2V^2}{b}.\frac{1}{(1+Z^2).Z^{2m-2}}$$

$$\times\left\{\frac{Z^{2m+1}-z^{2m+1}}{2m+1}+\frac{Z^{2m-1}-z^{2m-1}}{2m-1}\right\}, \qquad (5)$$

$$y=\frac{V^2}{b}.\frac{1}{(1+Z^2).Z^{2m-2}}$$

$$\times\left\{\frac{Z^{2m+2}-z^{2m+2}}{2m+2}-\frac{Z^{2m-2}-z^{2m-2}}{2m-2}\right\} \qquad (6)$$

$$s=\frac{V^2}{b}.\frac{1}{(1+Z^2)^2.Z^{2m-2}}$$

$$\times\left\{\frac{Z^{2m+2}+z^{2m+2}}{2m+2}+2.\frac{Z^{2m}-z^{2n}}{2m}+\frac{Z^{2m-2}-z^{2m-z}}{2m2}\right\} \qquad (7)$$

observing that $z=$ tan. $\frac{1}{2}u$, and $Z=$ tan. $\frac{1}{2}U$, u expressing generally the angle which the direction of the curve makes with the ordinate y.

From these expressions we may deduce several properties of this curve, as in the following Corollaries:

Cor. 1. As z universally decreases, and is always less than Z, it follows, that if $m>1$, the term z^{m-1} will become nothing, and v will vanish; so that in this case the body will come to a state of rest when $z=0$; the time of doing this is found in Cor. 12.

Cor. 2. If $m<1$, the term z^{m-1} becomes infinite when $z=0$; in this case the body never ceases to move, and finally acquires an infinite velocity on an infinitely extended plane.

Cor. 3. In the case of Cor. 1, where the body comes to a point of rest when $z=0$, the corresponding values of x, y, are,

$$x=\frac{2V^2}{b}.\frac{1}{(1+Z^2)^2}.\left\{\frac{Z^3}{2m+1}+\frac{Z}{2m-1}\right\} \qquad (8)$$

$$y=\frac{V^2}{b}\cdot\frac{1}{(1+Z^2)^2}\cdot\left\{\frac{Z^4}{2m+2}+\frac{1}{2m-2}\right\}. \qquad (9)$$

In this case y will be positive, if $Z^4>\frac{2m+2}{2m-2}$, otherwise negative; it will be nothing, if $Z^4=\frac{2m+2}{2m-2}$; and in this last case, the body will move till it arrives to the axis of x, when its motion will cease. This value of Z being greater than unity, proves that $\frac{1}{2}U>45°$, or $U>90°$, which was also very evident, because y would not be positive, unless $U>90°$.

Cor. 4. If $m=1$, we shall have $m=f\cot. e=1$, or $\tan. e=f$. If we suppose, as is usually done, that $f=\frac{1}{3}$, we shall have $e=18°\ 26'$. With this inclination the resistance of the plane is just equal to gravity, and $m=1$. In this case we shall have,

$$x=\frac{2V^2}{b}\cdot\frac{1}{(1+Z^2)^2}\cdot\left\{\frac{Z^3-z^3}{3}+\frac{Z-z}{1}\right\}, \qquad (10)$$

$$y=\frac{V^4}{b}\cdot\frac{1}{(1+Z^2)^2}\cdot\left\{\frac{Z^4-z^4}{4}-h.l.\frac{Z}{z}\right\}; \qquad (11)$$

because the integral of the term $\int dx\,z^{2m-3}$ which occurs in y, is generally $\frac{z^{2m-2}}{2m-2}$, and then becomes $\int\frac{dz}{z}=\log. z-\log. Z=\log.\frac{Z}{z}$. In this case y becomes infinite negative when $z=0$, and the greatest value of x then becomes

$$\frac{2V^2}{b}\cdot\frac{1}{(1+Z^2)^2}\cdot(\tfrac{1}{3}Z^2+Z):$$

and if to this value of x we draw the ordinate y, it will be the *asymptote* of the curve. The term Z^{m-1} in v, Eq. (4), will, in this case, become logarithmic, and its form may easily be found.

Cor. 5. If $m<1$, $>\frac{1}{2}$, the term z^{2m-1} of Eq. (5) vanishes when $z=0$, and the greatest value of x is finite, but the term z^{2m-2} of the Equation (6) becomes ∞, and y becomes ∞ negative when $z=0$; and in this case the ordinate drawn through the greatest value of x, will be an asymptote to the curve.

Cor. 6. If $m=\frac{1}{2}$ the terms $\frac{Z^{2m-1}z^{2m-1}}{2m-1}$ of x in Equation (5) arising from the integral of $-\int z^{2m-2}dz$ will become simply $-\int\frac{dz}{z}=-\log. z+\log. Z=\log.\frac{Z}{z}$, and x will become infinite as well as y, Equation (6).

Cor. 7. If $m<\frac{1}{2}$ the terms Z^{2m-1}, z^{2m-2}, Equations (5), (6) will be ∞ when $z=0$, consequently x, y will be infinite.

Cor. 8. If $m=0$, v Equation (4) becomes infinite when $z=0$; Equations (5), (6) give x infinite and y infinite negative. The particular nature of the curve in this case is mentioned in Cor. 12.

Cor. 9. Suppose the angle V to remain the same, but the velocity V to vary and become V', and let the terms x, y, s, v be accented, to denote their values corresponding to this new velocity; we shall have, in denoting for brevity by Z', Z'', Z''', &c. the functions of z, Z, by which V is multiplied in Equations (4), (7); then

$$v=Z'.V\ ,\ x=Z''.V^2\ ,\ y=Z'''.V^2\ ,\ s=Z^{IV}.V^2\ ,$$
$$v'=Z'.V'\ ,\ x'=Z''.V'^2\ ,\ y'=Z'''.V'^2\ ,\ s'=Z^{IV}.V'^2\ .$$

Hence we get $\frac{v'}{v}=\frac{V'}{V}$, and $\frac{x'}{x}=\frac{y'}{y}=\frac{s'}{s}=\frac{V'^2}{V^2}$; so that *the curves are similar*, and the homologous ordinates will be as V^2 to V'^2, or as v^2 to v'^2.

Cor. 10. The time t may also be expressed in terms of u or z, for $dt=\frac{ds}{v}$ gives $t=\int\frac{ds}{v}$, and by using the values of v, s in Eq. (4), (7),

$$t=\int\frac{ds}{v}=\int\frac{Z^{m-1}.(Z^2+1)}{V.z^{m-1}.(z^2+1)}+\frac{V^2}{b}.\frac{1}{(1+Z^2)^2.Z^{2m-2}}.dz$$
$$\left\{-z^{2m+1}-2z^{2m-1}-z^{2m-3}\right\}$$
$$=\frac{V}{b}.\frac{1}{(1+Z^2)Z^{m-1}}+\int dz\left(-z^m-z^{m-2}\right)$$
$$=\frac{V}{b}.\frac{1}{(4Z^2).Z^{m-1}}.\left\{\frac{Z^{m+1}-z^{m+1}}{m+1}+\frac{Z^{m-1}-z^{m-1}}{m-1}\right. \quad (12)$$

This value of t is generally greatest when $z=0$.

Cor. 11. If $m>1$ as in Cor. 12, the value of t becomes finite. If $m=1$, the term $\frac{z^{m-1}}{m-1}$ which arises from $\int z^{m-2}dz$, will depend on $\int\frac{dz}{z}$ or log. z, and being infinite, the time of cessation of motion will be infinite, as in Cor. 4. If $m<\frac{1}{2}$ the term, z^{m-1} becomes ∞ when $z=0$, and t is then infinite, as in Cor. 7.

Cor. 12. In the case treated of in Cor. 8, where $m=0$, if we put for brevity, $\frac{2V^2}{b}.\frac{1}{(1+Z^2)^2.Z^{-2}}=\frac{1}{A}$, the Equation (5) becomes $Ax+B=-z+z^{-1}$, and (6) becomes $4Ay+C=z^2-z^{-2}$, B and C being the functions of Z which occur in these Equations. The square of the former $(Ax+B)^2=z^2-2+z^{-2}$, adding this to the value of $4ay+C$, we get $(Ax+B^2+4Ay+C=-2$, the equation of a parabola, which is well known to be the curve described by a body moving without resistance.

Question XX.—OR PRIZE QUESTION.

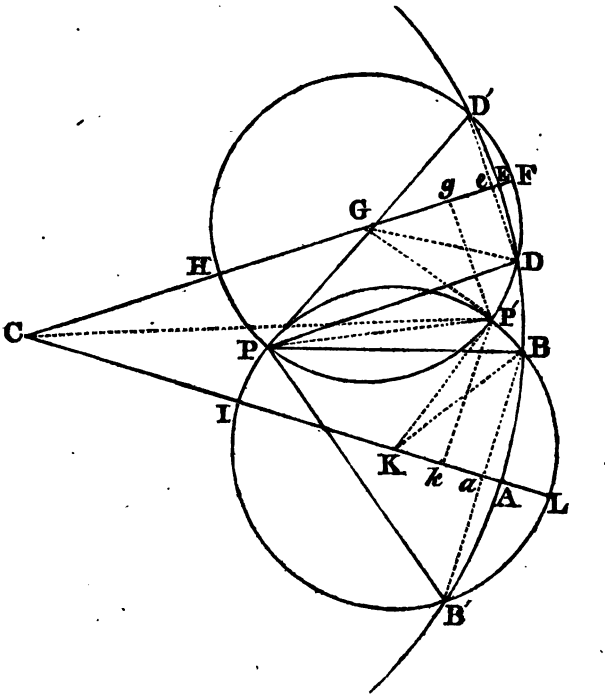
By Mr. P. Fleming, Civil Engineer and Surveyor, New-York.

If at a point within a given circle, the three angles be observed which are subtended by three contiguous arcs of the circumfer-

ence, of which the two extreme arcs are given in magnitude, but not in position; it is required to determine the magnitude of the intermediate arc, and the situation of the point where the observations are taken.

THE PRIZE SOLUTION.—*By Nathaniel Bowditch, LL.D. Boston.*

Let B′ A B D E D′ be the given circle, whose centre is C, P, the point of observation, and let B′ B D D′ be the proposed points of the given circle observed from P. Then we have given the angles B′ P B=b, B P D=b', D P D′=b'', B C B′=$2a$, D C D′=$2a''$, and the radius C A=ρ. Through the points



B B′ P, draw a circle whose centre is K, and also the circle D P D′, whose centre is G. Join GD, GP′, KB, KP′, DeD′, BaB′, and from the point P′, where the circles intersect, let fall on CL, CF the perpendiculars P′k, P′g; and put CK=R, CG=R″, KB=r, GD=r'', all of which will be given.

In the triangle BKa we have Ba=r . sine b, and in the triangle BCa, Ba=ρ . sine a; hence r . sine b=ρ . sine a, $\therefore r=\frac{\rho \cdot \text{sine } a}{\text{sine } b}$, and the same triangles give Ka=r . cos. b, Ca=ρ . cos. a $\therefore$ CK=Ca—Ka=ρ . cos. a—r . cos. b, or

$$R=\rho \cdot \cos. a-\frac{\rho \cdot \text{sine } a}{\text{sine } b} \cdot \cos. b=\rho\,(\cos. a-\text{sine } a \cdot \cot. b).$$

In like manner the triangles eGD, eCD, give

$$r''=\rho \cdot \frac{\text{sine } a''}{\text{sine } b''} \text{ and } R''=\rho \cdot (\cos. a''-\text{sine } a'' \cdot \cot. b'').$$

Again: the angles B'KB=2B'PB=2b, BKP'+P'GD=2BPP'+2P'PD=2BPD=2b', DGD'=2DPD'=2b''; the sum of these three expressions is (B'KB+BKP')+(P'GD+DG'D), or more simply, B'KP'+P'GD'=2b+2b'+2b'': subtracting BKA=b and D'GD=b'', we get

$$LKP'+FGP'=b+2b'+b''=2B,$$

using 2B for brevity, instead of b+2b'+b''. Now the sum of the two angles LKP', FGP' being equal to 2B, we shall put LKP'=B+x, FGP'=B—x, and thus in the triangle KPk we shall get

$$P'k=r \cdot \text{sine } (B+x),\ Kk=r \cdot \cos. B+x,$$

whence Ck=R+r . cos. (B+x); consequently $CP'^2=Ck^2+P'k^2=\{R+r \cdot \cos. (B+x)\}^2+r^2 \cdot \text{sine}^2(B+x)=R^2+r^2+2Rr \cdot \cos. (B+x)$.

In like manner, in the triangle GgP' we get P'g=r'' . sine (B—x), Gg=r'' . cos. (B—x), eg=R''+r'' . cos. (B—x), and $CP'^2=Cg^2+P'g^2=R''^2+r''^2+2R''r'' \cdot \cos. (B-x)$.

Equating these values of CP'^{2}, we get

$$R''^2+r''^2-R^2-r^2=2Rr \cdot \cos. (B+x)-2R''r'' \cdot \cos. (B-x);$$

or, by developing the cosines, $R''^2+r''^2-R^2-r^2=(2Rr-2R''r'')\cos. B \cos. x-(2Rr+2R''r'')\text{ sine B sine } x$.

If we divide this by the (2Rr—2R''r'') cos. B, and for brevity put $\frac{R''^2+r''^2-R^2-r^2}{(2Rr-2R''r'')\cos. B}=D$, and $\frac{Rr+R''r''}{Rr-R''r''}\tan. B=\tan. C$, it becomes $D=\cos. x+\tan. C \cdot \text{sine } x=\cos. x+\frac{\text{sine C}}{\cos. C} \cdot \text{sine } x=\cos.$

$x \cos. C+\text{sine } x \text{ sine C}=\frac{\cos. (x-C)}{\cos. C}$, whence $\cos. (x-C)=D \cdot \cos. C$: and as C, D are given, we easily deduce the value of $x-C=\pm m$, m being the tabular arch whose cosine is D . cos. C; whence LKP'=B+x, FGD'=B—x, from which we get P'g, P'k, Cg, Ck, and the angles P'CG, P'CA, whose sum gives the whole angle ECA, and by subtracting the angles ACB=a, ECD=a'', we get the angle BCD, or the required arch BD.

In this solution I have taken the point P, which is nearer the centre C than P' is; the reasoning would have been nearly the same, if we had used the point P' instead of P; but it is not necessary to go into the detail of the cases.

ACKNOWLEDGMENTS, &c.

THE following ingenious gentlemen favoured the Editor with solutions to the questions proposed in No. 1. Article III. The figures annexed to the names, refer to questions answered by each, as numbered in that article.

Nathaniel Bowditch, LL. D. of Boston, and Professor Strong, Hamilton College, Oneida County, State of New-York, each most ingeniously answered all the Questions.

Farrand N. Benedict, Montezuma, N. Y. State,								5					11									
P. Byrne, Philadelphia,		2								8												
John Capp, Esq. Harrisburg, Penn.	1	2			5	6	7				10											20
Denis Carmody, New-York,	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		17	18		20
Rev. T. Clowes, Chestertown, Maryland,	1	2	3	4	5	6	7	8	9	10												
Henry Darnall, Philadelphia,	1	2	3	4	5	6	7	8	9	10	11							18				
Rev. Henry Doyle, New-York	1	2	3	4														18				
James O. Farrell, New-York,	1																					
Peter Fleming, Do.				4						10							17	18		20		
B. McGowan, Do.	1	2	3	4	5	6	7	8	9	10		12			15	16	17	18		20		
Denis Leonard, Do.		2	3		5		7											18				
Walter Patterson, Do.	1							8														
James Phillips, Harlem,	1	2	3	4	5	6	7	8	9	10										20		
Mr. Quinby, New-York,																		18				
Martin Roche, Philadelphia,			3	4	5	6	7			10												
John Rochford, New-York,	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16				20		
Daniel Shanley, Charleston, S. C.		2	3	4		6	7															
Seth Smith and John H. Willets, Philadelphia,	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18		20		
Joseph Strode, Chester Co. Penn.	1	2	2	4	5																	
Tyro, of Lexington,	1	2	3	4	5	6	7				11											
Farrell Ward, New-York,	1	2	3	4	5	6	7	8	9	10					15							
Charles Wilder, Baltimore,	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18		20		

The Prize for the best solution of the Prize Question No. 20, is awarded to Nathaniel Bowditch, LL. D. of Boston. Several other ingenious solutions were received, particularly those of Messrs. Capp, Fleming, and Phillips.

ARTICLE V.

NEW QUESTIONS,

TO BE RESOLVED BY CORRESPONDENTS IN NO. III.

QUESTION I. (21.)—*By Mr. Dennis Leonard, New-York.*

THERE are three numbers in geometrical progression, the difference of whose differences is 6, and their sum is 42. Quere, the numbers?

QUESTION II. (22.)—*By Mr. Daniel Shanley, Charleston, S. C.*

There are three numbers in harmonical proportion, the difference of whose differences is 2, and four times the product of the first and third is 288. Required the numbers.

QUESTION III. (23.)—*By Mr. James Phillips, Harlem, near New-York.*

It is required to find three numbers having equal differences, and such, that if the first be increased by 1, the second by 2, and the third by the first, the sums may constitute an harmonic progression; but if 3 be added to the second, it may be a mean proportional between the sum of the numbers and the first, diminished by $\frac{1}{2}$.

QUESTION IV. (24.)—*By Mr. Peter Fleming, New-York.*

Given the three following equations,

$$x^2y\sqrt{xy} = a,$$

$$xz\sqrt{xz^3} = b,$$

$$yz\sqrt{xy^3} = c,$$

to find the values of x, y, and z.

QUESTION V. (25.)—*By Mr. Thomas Brady, New-York.*

Given the following equations,

$$\sqrt{x}+\sqrt{x-2}=\frac{2y^2}{x-2},$$

$$\frac{y-\sqrt{y^2-x^2}}{y+\sqrt{y^2-x^2}}=\frac{y}{x},$$

to find the values of x and y.

QUESTION VI. (26.)—*By John Capp, Esq. of Harrisburg, Pa.*

To find two squares, x^2 and z^2, such, that their difference be to the root of the lesser square in the ratio of 3 to 7, and that the roots of the two squares be to each other in the ratio of 5 to 2.

QUESTION VII. (27.)—*By Mr. Denis Carmody, New-York.*

It is required to find four numbers in arithmetical progression, such that the sum of their squares shall be a square, and the product of the extremes increased by the product of the means, shall be a cube number, with the investigation.

QUESTION VII. (28.)—*By Eboracensis.*

To find two numbers such that if each be added to the reciprocal of the other, the two sums may both be squares.

QUESTION IX. (29.)—*By Diophantus.*

To find a right angled triangle, such that the area increased by a given number, (5) may be a square.

N. B. This is the third question of the sixth book of Diophantus's celebrated treatise of algebra; his second question of the same book, differing from the first only, in having the word *increased*, instead of the word *diminished*.

QUESTION X. (30.)—*By Mr. Farrell Ward, New-York.*

If an arc of a circle be bisected, and produced to any point, the rectangle contained by the sine of the whole arc thus produced, and the sine of the part produced together with the square of the sine of half the arc bisected, is equal to the square of the sine of the arc made up of the half and the part produced; a geometrical demonstration is required.

QUESTION XI. (31.)—*By Mr. Edward Ward, New-York.*

Having a field in form of a trapezoid, whose parallel sides are 500 and 300 yards respectively, and at right angles to one of the remaining sides; the perpendicular distance from the intersection of the diagonals to the less parallel side is 250 yards, the distance between the parallel sides is required.

QUESTION XII. (32.)—*By Tyro, Lexington, Kentucky.*

If a glass in the form of an inverted cone 6 inches high, and 5 inches in the diameter of its base be $\frac{1}{8}$ part filled with water, what must be the dimensions of a cylinder which being immersed in the water in the direction of its axis, shall raise the water to the greatest height possible.

QUESTION XIII. (33.)—*By Patterson Wallace, New-York.*

Given the sum of the two sides of a right angled plane triangle, and the difference between the hypothenuse and the perpendicular let fall on it from the right angle, to determine the sides of the triangle.

QUESTION XIV. (34.)—*By Mr. Philip W. Hanson, New-York.*

Given the hypothenuse 60°, and the sum of the legs 70° of a right angled spherical triangle, to determine the legs of the triangle.

QUESTION XV. (35.)—*By Eboracensis.*

To inscribe the greatest rectangle in one of the ovals of a lemniscate of which the equation is

$$a^4y^2=a^4x^2-x^6$$

QUESTION XVI. (36.)—*By Mr. A. B. Quinby, New-York.*

If an infinitely small straight rod, that is suspended at one of its extremities, be raised till it form any given angle with the vertical, and then be allowed to descend, it is required to determine the position, it will have, when the quantity of its motion that it will have in a vertical direction is a maximum.

QUESTION XVII. (37.)—*By Mr. John Roehford, New-York.*

Two ships set sail at the same instant from Sligo for New-York, one in the arc of a great circle, the other on the rhumb line, connecting both cities, and keep always due north and south of each other, required their latitudes and longitude when they are the greatest possible distance asunder.

QUESTION XVIII. (38.)—**OR PRIZE QUESTION.**

By Professor Adrain, Columbia College, New-York.

It is required to investigate the path that ought to be described by a boat in crossing a river of given breadth from a given point on one side, to a given point on the other, so as to make the passage in the least time possible: supposing the simple velocity of the boat by the propelling power to be given; and that the velocity of the current, being in the same direction with the parallel sides of the river, is variable and expressed by any given function of the perpendicular distance from that side of the river from which the boat sets out.

☞ The Editor regrets his being obliged to suppress a great number of ingenious solutions on account of the narrow limits of the Diary, which could not contain more than half the number of good solutions that were received.

THE

MATHEMATICAL DIARY,

N° III.

BEING THE PRIZE NUMBER OF MR. T. STRONG, PROFESSOR OF MATHEMATICS IN HAMILTON COLLEGE, ONEIDA COUNTY, STATE OF N. YORK.

ARTICLE VI.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE V. NO. II.

QUESTION I.—*By Mr. Dennis Leonard, N. York.*

THERE are three numbers in geometrical progression, the difference of whose differences is 6, and their sum is 42. Quere, the numbers?

FIRST SOLUTION.—*By Mr. Bogart, N. York.*

Let x, y and z be taken the three required numbers, and agreeably to the conditions of the question we have

$(x-y)-(y-z)=x-2y+z=6$, and $x+y+z=42$;

the difference of which two equations divided by 3 gives $y=12$. Then $x+z=42-y=30$, and $xz=y^2=144$, therefore $(x+z)^2-4xz=30^2-4\times144=324$, whence $x-z=18$: now by taking the sum and difference of $x+z=30$, and $x-z=18$, we have $x=24$, $z=6$, and therefore 24, 12, 6, are the required numbers.

SECOND SOLUTION.—*By Mr. Ingersoll Bowditch, Boston.*

Let x, y, z be the numbers, then we have per question $x:y::y:z$, or $y^2=xz$, also $x-2y+z=6$, and $x+y+z=42$, subtracting the former equation from the latter, we get $3y=36$, or $y=12$. From the equation $y^2=xz$, we have $x=\frac{144}{z}$, and substituting the values we have $\frac{144}{z}+12+z=42$, whence $z^2-30z=-144$, by completing the square and extracting the root we obtain $z-15=9$ or $z=24$, and from the equation $x=\frac{144}{z}$ we find $x=6$, therefore the required numbers are 6, 12, 24.

THIRD SOLUTION.—*By Mr. James Hamilton, Trenton.*

Let $x=$ the less and $a=$ the ratio, then by the question

$$a^2x+ax+x=42$$
$$\text{and } a^2x-2ax+x=6$$

By subtraction $3ax=36$, and $x=\frac{12}{a}$, then by substituting $12a+12+\frac{12}{a}=42$, which is reducible to $a^2-\frac{5a}{2}=-1$, whence $a=2$, and the numbers are 24, 12, 6.

Two or three ingenious correspondents understood the words, "their sum," in the question, to signify the sum of the differences, and, according to this interpretation, resolved the problem in the following manner.

FOURTH SOLUTION.—*By Mr. Walter Nichols, N. York.*

Let $x=$ the last number and ρ the ratio, then by hypothesis

$$\rho^2x-2\rho x+x=6,$$
$$\rho^2x \quad - \quad x=42.$$

By addition and dividing by 2, we have $\rho^2x-\rho x=24$,
and by subtraction and dividing by 2 $\rho x-x=18$,
the last of which, dividing the last but one, gives $\rho=\frac{4}{3}$, whence the numbers are 96, 72, 54.

QUESTION II.—*By Mr. Daniel Shanley, Charleston, S. C.*

There are three numbers in harmonical proportion, the difference of whose differences is 2, and 4 times the product of the first and third is 288. Required the numbers.

FIRST SOLUTION.—*By Mr. Phillips, Harlem.*

Let $x=$ the first or least number, and $y=$ difference between the first and second, then x, $x+y$, $x+2y+2$ are the numbers, therefore by harmonic proportion we have

$$x : x+2y+2 :: y : y+2,$$

and disjunctly $2(y+1) : x :: 2 : y$, whence $x=(y+1)y$.

Again $x^2+2xy+2x=x^2+2(y+1)x=72$, in which putting for x its value $(y+1)y$ we have $(y+1)^4-(y+1)^2=72$; from which quadratic $y=2$, and consequently, 6, 8, and 12 are the numbers required.

SECOND SOLUTION.—*By Mr. Nicholas Donnelly, N. Y.*

Let x, y, z, be the required numbers, then by the question we have $x-2y+z=2$, $4xz=288$, and $x : z :: z-y : y-z$; from the last by multiplication $(x+z)2y=4xz=288$, and by division $x=\frac{144-yz}{y}$; but from the first equation $z=2+2y-x$, therefore $2+2y-z=\frac{144-yz}{y}$, which cleared of fractions is $2y+2y^2-yz=144-yz$, therefore $y^3+y=72$, and $y=8$. Again $x=\frac{144-yz}{y}=18-$

z, which value, if x being substituted in $4xz=288$, there results $z^2-18z=-72$, and $z=6$, whence $x=18-6=11$, and the numbers are 12, 8, 6.

THIRD SOLUTION.—*By Mr. Henry Darnall, Philadelphia.*

Let x, y, z, represent the three required numbers, then by harmonic proportion $x:z::x-y:y-z$, from which $y=\frac{2xz}{x+z}=\frac{144}{x+z}$, because $4xz=288$. Again $x+z-2y=2$ per ques. from which $y=\frac{x+z-2}{2}=\frac{144}{x+z}$, whence $(x+z)^2-2(x+z)=288$; this quadratic gives $x+z=18$, and $x=18-z=\frac{72}{z}$, whence $18z-z^2=72$, and $z=12$ or 6, and the numbers are 12, 8, 6.

QUESTION III.—*By Mr. James Phillips, Harlem, near N. Y.*

It is required to find three numbers having equal differences, and such that if the first be increased by 1, the second by 2, and the third by the first, the sums may constitute an harmonic progression; but if 3 be added to the second, it may be a mean proportional between the sum of the numbers, and the first diminished by $\frac{1}{2}$.

FIRST SOLUTION.—*By the Rev. Dr. Clowes, Chestertown, Md.*

Let $x-d$, x, $x+d$ be the numbers; then $x-d+1$, $x+2$ and $2x$ are in harmonic proportion, whence $d=\frac{x^2-3x+2}{3x-2}$. Then per ques. $3x(x-\frac{x^2-3x+2}{3x-2}-\frac{1}{2})=(x+3)^2$. This equation reduced we have $x^3-\frac{35}{6}.x^2-2x+6=0$, whence $x=6$, and $d=1$, then $x-d$, x, $x+d$ are 5, 6, 7, the numbers required.

SECOND SOLUTION.—*By Mr. Michael Floy, N. Y.*

Let $x+y$, x and $x-y$ represent any three numbers in arithmetical proportion, then by the question

$$x+y+1:2x::y-1:2-x,$$

and

$$3x(x+y-\tfrac{1}{2})=(x+3)^2,$$

whence by reduction we have the two equations,

$$x^2+3xy-3x-2y=2,$$

$$4x^2+6xy-15x=18.$$

From these $y=\frac{18-4x^2+15x}{6x}$, $y=\frac{2+3x-4x^2}{3x-2}$,

consequently $\frac{18+15x-4x^2}{6x}=\frac{2+3x-x^2}{3x-2}$,

which reduced becomes

$$6x^3-35x^2-12x+36=0.$$

From this cubic we find $x=6$, whence $y=-1$, and therefore $x+y$, x, $x-y$, are 5, 6, 7, the numbers sought.

THIRD SOLUTION.—*By Mr. Walter Nichols, N. Y.*

Let x, y, z be the numbers, and per question $x+z=2y$, and $x+1 : 2y :: y-x+1 : y-2$, whence $x=\frac{2y^2+y+2}{3y-2}$.

Again by the question $x-\frac{1}{2} : y+3 :: y+3 :$ from which is had $x=\frac{y^2+\frac{15}{2}y+9}{3y}$ therefore $\frac{y^2+\frac{15}{2}y+9}{3y}=\frac{2y^2+y+2}{3y-2}$, which equation reduced is $y^3-\frac{35}{6}y^2-2y+6=0$, hence $y=6$. whence reduced numbers are 5, 6, 7.

The solutions of Mr. F. Ward and Mr. Wilder were different in their notation from any of the preceding three. They assumed $x-1$, $y-2$, and $2y-x-3$ for the numbers sought.

QUESTION IV.—*By Mr. Peter Fleming, New-York.*

Given the three following equations,

$x^2y\sqrt{xy}=a$, $xz\sqrt{xz^3}=b$, $yz\sqrt{zy^3}=c$, to find the values of x, y, and z.

FIRST SOLUTION.—*By Mr. Wm. Lenhart, Yorktown, Penn.*

Clearing the three given equations of surds we have $x^5y^3=a^2$, $x^3z^5=b^2$, $y^5z^3=c^2$. Let now $x=vz$, $y=wz$, and by substitution $v^5w^3z^8=a^2$, $v^3z^8=b^2$, $w^5z^8=c^2$, and dividing the first by the second and the third, we find $v^2w^3=\frac{a^2}{b^2}$ and $\frac{v^5}{w^2}=\frac{a^2}{c^2}$. From the first of these we have $w^6=\frac{a^4}{b^4v^4}$, and from the second, we have $w^6=\frac{c^6a^{15}}{a^6}$, therefore $\frac{c^5v^{15}}{a^6}=\frac{a^4}{b^3v^4}$, whence $v^{19}=\frac{a^{10}}{b^4c^6}$, and $v=\sqrt[19]{\frac{a^{10}}{b^4c^6}}$, from which the numbers sought are easily found.

SECOND SOLUTION.—*By Mr. Charles Farquhar, Alexandria, District of Columbia.*

Multiplying the three given equations together, there results $xyz=\sqrt[4]{abc}=p$, thence $xy=\frac{p}{z}$ and $x^2y=\frac{px}{z}$, substitute these

in the first equation and $\frac{px}{z}\sqrt{\frac{p}{z}}=a$: again, from the second equation $x^3z^5=b^2$, or $x=\sqrt[3]{\frac{b^2}{z^5}}$, then by substitution in the above value of a, $\sqrt[3]{\frac{b^2p^3}{z^8}}\cdot\sqrt{\frac{p}{z}}=a$, or by squaring, $\frac{p}{z}\sqrt[3]{\frac{b^4p^6}{z^{16}}}=a^2$, and cubing this, $\frac{b^4p^9}{z^{19}}=a^6$, or $z=\left(\frac{b^4p^9}{a^6}\right)^{\frac{1}{19}}$, and from these the values of x and y are easily found.

Several correspondents remarked the application of logarithms to the solution of this question, the results, according to Mr. Wilder and the Rev. Dr. Clowes, are

$$\text{Log. } x=\frac{25 \text{ log. } a+9 \text{ log. } b-15 \text{ log. } c}{76},$$

$$\text{Log. } y=\frac{9 \text{ log. } a+25 \text{ log. } c-15 \text{ log. } b}{76},$$

$$\text{Log. } z=\frac{25 \text{ log. } 6+9 \text{ log. } c-15 \text{ log. } a.}{76}$$

QUESTION V.—*By Thomas Brady, New-York.*

Given the following equations, $\sqrt{x}+\sqrt{x-2}=\frac{2y^2}{x-2}$

$$\frac{y-\sqrt{y^2-x^2}}{y+\sqrt{y^2-x^2}}=y^2,$$

to find the values of x and y.

FIRST SOLUTION.—*By Mr. Phillips, Harlem.*

The second equation becomes, by multiplication, $y-\sqrt{y^2-x^2}=y^3+y^2\sqrt{y^2-x^2}$ hence $\sqrt{y^2-x^2}=\frac{1-y^2}{1+y^2}\cdot y$, from which we obtain $x=\frac{2y^2}{1+y^2}$: now substitute this value of x in the first equation gives $\left(\frac{2y^2}{1+y^2}\right)^{\frac{1}{2}}+\left(-\frac{2}{1+y^2}\right)^{\frac{1}{2}}=y^2(1+y^2)$, which by an obvious reduction becomes

$$y^{12}+2y^8+20y^6+33y^4+20y^2+4=0,$$

some of the roots of which will satisfy the two original equations.

SECOND SOLUTION.—*By Mr. Wm. Lenhart, Yorktown, Penn.*

Multiply the fraction in the second equation above and below by $y+\sqrt{y^2-x^2}$ and we have $\frac{y^2-y^2+x^2}{(y+\sqrt{y^2-x^2})^2}=y^2$, which, by reduction gives $y^2=\frac{x}{2-x}$.

Let now $x=\left(\frac{a^2+2}{2a}\right)^2$, and by substitution in the first equation we shall find $\sqrt{x+\sqrt{x-2}}=a=\frac{2y^2}{x+2}=\frac{2x}{(2-x)(x-2)}=\frac{a^4+4a^2+4}{2a^2}\times\frac{16a^4}{4a^2-a^4-4.(a^2-2)^2}$, or by reduction,

$$a^8-8a^6+24a^4+32a^3-32a^2+32a+26=0.$$

from which equation a may be found.

By some inadvertency the equations in this 5th question were different from those of the proposer: his equations were

$$\frac{\sqrt{x+\sqrt{x-2}}}{\sqrt{x-\sqrt{x-2}}}=\frac{2y^2}{x-2},$$

$$\frac{y-\sqrt{y^2-x^2}}{y+\sqrt{y-x^2}}=y^2.$$

QUESTION VI.—*By John Capp, Esq. Harrisburg, Pa.*

To find two squares, x^2 and z^2, such, that their difference be to the root of the lesser square in the ratio of 3 to 7, and that the roots of the two squares be to each other in the ratio of 5 to 2.

SOLUTION.—*By Mr. James Quin, New-York.*

By the question $x^2-z^2 : z :: 3 : 7$. and $x : z :: 5 : 2$, from which we obtain $7x^2-7z^2=3z$, and $2x=5z$, from which last $x=\frac{5z}{2}$: this value of x being substituted in the first equation gives $\frac{175z^2}{4}-7z^2=3z$, or multiplying by 4, $175z^2-28z^2=12z$, or $147z^2=12z$, and dividing by z, $147z=12$, $z=\frac{12}{147}=\frac{4}{49}$ and $x=\frac{5z}{2}=\frac{10}{49}$, whence the numbers required are $\left(\frac{10}{49}\right)^2$ and $\left(\frac{4}{49}\right)^2$.

This solution may with equal propriety be ascribed to Mr. Vogdes, Edgmont, Delaware Co. Pa.; Mr. William F. Kells, Bergen, New-Jersey; Mr. Charles Potts, Philadelphia; Mr. Ingersoll Bowditch of Boston, and Samuel Bard McVickar, N. York.

QUESTION VII.—*By Mr. Denis Carmody, N. York.*

It is required to find four numbers in arithmetical progression, such that the sum of their squares shall be a square, and the product of the extremes increased by the product of the means, shall be a cube number, with the investigation.

FIRST SOLUTION.—*By John Capp, Esq. Harrisburg, Pa.*

Let the first of the required numbers be denoted by x, and their common difference by y, then the numbers will be represented by x, $x+y$, $x+2y$, $x+3y$, the sum of their squares by $4x^2+12xy+14y^2$, which must be a square, and the sum of the products of the extremes and means by $2x^2+6xy+2y^2$, which must be a cube. By assuming $y=vx$, our first formula becomes $(4+12v+14v^2)x^2=\square$, and the second $(2+6v+2v^2)x^2=$ cube.

Assuming $4+12v+14v^2=(rv-2)^2=r^2v^2-4rv+4$, we have $v=\frac{4r+12}{r^2-14}$. By taking $r=4$, we obtain $v=14$, and our second formula $(2+6v+2v^2)x^2=478x^2$, which is evidently a cube when $x=478$, and consequently $y=vx=6692$; hence the numbers will be 478, 7170, 13862, 20554. We may obtain lower numbers by assuming a different value for r. If r be taken $=7$, we get $v=\frac{3}{7}$, and by proceeding as above we obtain 3934, 8430, 12926, 17422, for the numbers.

SECOND SOLUTION.—*By Mr. C. Wilder, Baltimore.*

If we make $qa=a\frac{p^2-14}{4p+12}$, the numbers qa, $qa+a$, $qa+2a$, $qa+3a$ will fulfil the first two conditions of the question: and if we make $r^3a^3=2q^2a^2+6qa^2+a^2$, we have $a=\frac{2q^2+6q+2}{r^3}$. Example, taking $p=4$, and $r=\frac{1}{14}$ we have $9=\frac{1}{14}$, and $a=5092$, then the number will be 478, 7170, 13862, 20554.

THIRD SOLUTION.—*By Mr. Joseph C. Strode, Philadelphia.*

Let x, $x+y$, $x+2y$, $x+3y$ be the numbers, the sum of their squares $4x^2+12xy+14y^2$ must be a square; assume its root $=2x-ny$, and by reduction we find $y=\frac{4n+12}{n^2-14}\cdot x$; there only remains to make the product of the extremes increased by the product of the means a cube, which by substitution is

$$\frac{2n^4+24n^3+48n^2-144n-328}{(n^2-14)^2}\cdot x^2,$$

this will evidently be a cube when

$$x=\frac{2n^4+24n^3+48n^2-144n-328}{(n^2-14)}.$$

Let the value of x be denoted by p, and put $\frac{4n+12}{n^2-14}=q$, and the four numbers become p, $p(q+1)$, $p(2q+1)$, $p(3q+1)$.

If $n=4$, then $q=14$, $p=478$, and the numbers are 478, 7170, 13862, 20554.

QUESTION VII.—*By Eboracensis.*

To find two numbers such that if each be added to the reciprocal of the other, the two sums may both be squares.

FIRST SOLUTION.—*By Mr. Farrel Ward, New-York.*

Let x^2 and x^2+2 be the numbers, then $x^2+2+\frac{1}{x^2}$ is evidently a square; and $x^2+\frac{1}{x^2+2}=\frac{x^4+2x^2+1}{x^2+2}$ must be a square, it only remains therefore to make x^2+2 a square: assume $x^2+2=(x-n)^2$, hence $x=\frac{n^2-2}{2n}$, where n may be taken at pleasure. If $n=2$, $x=\frac{1}{2}$ and the numbers are $\frac{1}{4}$, $\frac{9}{4}$.

SECOND SOLUTION.—*By Aliquis, Herkimer co. N. York.*

Let $16x^2$ and $\frac{1}{9x^2}$ be the numbers, and all the conditions are answered by the assumption: if $x=1$, we have 16 and $\frac{1}{9}$ for the numbers, &c.

THIRD SOLUTION.—*By Mr. F. Benedict, Montezuma, N. Y.*

Denoting the numbers sought by $\frac{1}{2ax+a^2}$ and x^2, the conditions of the question require that $x^2+2ax+a^2=$ a square and $\frac{1}{2ax+x^2}+\frac{1}{x^2}=\frac{x^2+2ax+x^2}{2ax^3+a^2x^2}=$ a square. If we assume $2ax^2+a^2x^2=m^2x^2$ we have $x=\frac{m^2-a^2}{2a}$; if $m=6$, $a=2$, then 64 and $\frac{1}{36}$ are the numbers sought.

FOURTH SOLUTION.—*By Mr. Hamilton, Trenton.*

Let x^2 and $\frac{1}{2x+1}$ be the number, then x^2+2x+1 is a square, and we have only to make $\frac{1}{x^2}+\frac{1}{2x+1}=\frac{x^2+2x+1}{2x^3+x^2}=\square$; put

$2x^3+x^2=n^2x^2$, and $x=\frac{n^2-1}{2}$. Let $n=2$, then $x=\frac{3}{2}$ and the numbers are $\frac{9}{4}$ and $\frac{1}{4}$.

FIFTH SOLUTION.—*By the Rev. Dr. Clowes, Chestertown, Md.*

Let x^2 and y^2 be the two numbers, then $x^2+\frac{1}{y2}$, and $y^2+\frac{1}{x2}$ must be squares. Put $x^2+\frac{1}{y2}=(x-\frac{p}{y})^2$, whence $x=\frac{p^2-1}{2py}$.

Put $p=2$, $y=3$ and we have $x=4$, hence the numbers required are 9 and $\frac{1}{16}$.

SIXTH SOLUTION.—*By Mr. Lenhart, Yorktown, Pa.*

Let x and y represent the two numbers, then by the question $x+\frac{1}{y}$ and $y+\frac{1}{x}$ are to be made squares. Put $x+\frac{1}{y}=n^2$ whence $y=\frac{1}{n^2-x}$. In the formula $y+\frac{1}{x}$ substitute for y its value $\frac{1}{n^2-x}$ and we have $\frac{1}{n^2-x}+\frac{1}{x}=\square$, or multiplying by x^2, we must have $\frac{x^2}{n^2-x}+x=\square=m^2$ whence $x=\frac{m^2n^2}{m^2+n^2}$. If $m=1$, $n=1$, then $x=\frac{1}{2}$ and $y=2$.

QUESTION IX.—*By Diophantus.*

To find a right angled triangle, such that the area increased by a given number, (5) may be a square.

SOLUTION.—*By Mr. Capp, Harrisburg.*

Let $(x^2+y^2)n$, $(x^2-y^2)n$, $2nxy$ be the sides of the triangle, then the area will be $(x^2-y^2).n^2xy$, and $(x^2-y^2).n^2xy=$ a square; to effect this take $x=5$, $y=4$, then $x^2-y^2=9$, and $(x^2-y^2)n^2xy+5=20\times9n^2+5=$ a square, which is evidently the case if $9n^2=1$, or $x=\frac{1}{3}$, hence the sides of the triangle are $\frac{40}{3}$, $\frac{41}{3}$, $\frac{9}{3}$.

This question which was originally proposed and resolved by Diophantus, was afterwards resolved by Vieta, in the 9th problem of the 5th book of his Zetetics; his method is applicable to any case in which the given number as 5 is the sum of two squares.

Prestet resolved the same problem in his Elements of Mathematics; and Kersey, in his Elements of Algebra, resolves the

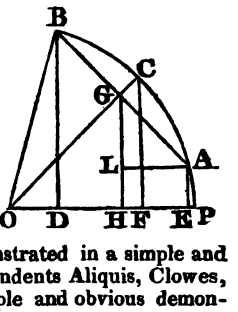
problem in an ingenious, explicit, and general manner; he has not, however, detected such simple numbers as those given by Mr. Capp in the preceding solution; the numbers given by Mr. Capp were also obtained by Mr. F. Ward, and Mr. B. McGowan.

QUESTION X.—*By Mr. Farrel Ward, New-York.*

If an arc of a circle be bisected, and produced to any point, the rectangle contained by the sine of the whole arc thus produced, and the sine of the part produced together with the square of the sine of half the arc bisected, is equal to the square of the sine of the arc made up of the half and the part produced; a geometrical demonstration is required.

SOLUTION.—*By Mr. Charles Farquhar, Alexandria, D. C.*

Let BA be the arc bisected in C and produced to P, then it is to be shown that $BD.AE+AG^2=CF^2$. By simtri. CFO, ALG, $CF^2:LA^2::CG^2:AG^2$, and by division $CF^2-LA^2:LA^2::OG^2:AG^2::HG^2:LA^2$, therefore $CF^2-LA^2=HG^2$, or $CF^2=HG^2+LA^2=(AE+LG)^2+LA^2=AE(AE+2LG)+LG^2+LA^2=AE.BD+AG^2$.



This curious theorem was also demonstrated in a simple and elegant manner by our ingenious correspondents Aliquis, Clowes, and the proposer, F. Ward. A very simple and obvious demonstration may also be deduced from Ptolemy's theorem of a quadrilateral inscribed in a circle, by producing the sines AE, CF, BD until they meet the circumference; and then a trapezoid will be inscribed in the circle of which each diagonal will be the double of CF. Then by Ptolemy's theorem

$$2AE.2BD+AB^2=(2CF)^2$$

which divided by 4 gives

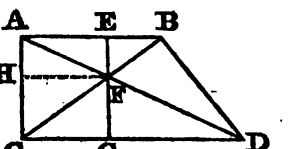
$$AE.BD+AG^2=CF^2.$$

QUESTION XI.—*By Mr. Edward Ward, New-York.*

Having a field in form of a trapezoid, whose parallel sides are 500 and 300 yards respectively, and at right angles to one of the remaining sides; the perpendicular distance from the intersection of the diagonals to the less parallel side is 250 yards, the distance between the parallel sides is required.

FIRST SOLUTION.—*By the Rev. H. Doyle, New-York.*

The diagonals form with the parallel sides two similar triangles ABF, CDF, whose bases CD, and AB are as their respective perpendiculars FG and EF: we have, therefore, the stating as AB : CD :: EF : FG, that is as 300 : 500 : : 250 : FG = 315 . 66 &c. which added to EF=250 gives E G=666 . 66 &c.



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SECOND SOLUTION.—*By Mr. Daniel D. Aiken, Quaker Hill, N. Y.*

Let $x+250$=the distance required E G (see fig. to preceding solution) between the parallel sides A B, C D, then by similar triangles $x : 500 : : 250 : 300$, whence $300x=125000$ and $x=\frac{1250}{3}=416,66$, and $x+250=666,66$=the distance required.

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THIRD SOLUTION.—*By Mr. James M'Ginnis, Harrisburg, Pa.*

Put CD=500=a, AB=300=c, EF=AH=250=b, CG=HF =x, GD=$a-x$, GF=y. Then by sim. triangles $a-x : y : : x : b=\frac{xy}{a-x}$, whence $y=ab-bx$.

Again by sim. tri. $c-x : b : : x : y=\frac{bx}{c-x}$; and comparing the values of y we have $\frac{ab-bx}{x}=\frac{bx}{c-x}$, whence $x=\frac{ac}{a+c}=666\frac{2}{3}$=the distance required.

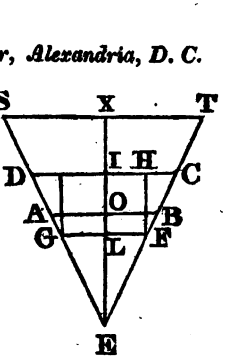
This question was also resolved in a simple and easy manner by Mr. Dennis Leonard, New-York.

QUESTION XII.—*By Tyro, Lexington, Kentucky.*

If a glass, in form of an inverted cone, 6 inches high and 5 inches in the diameter of its base, be $\frac{1}{8}$ part filled with water, what must be the dimensions of a cylinder which being immersed in the water in the direction of its axis, shall raise the water to the greatest height possible.

FIRST SOLUTION.—*By Mr. C. Farquhar, Alexandria, D. C.*

Let SET be a section of the glass through the axis EX, and let AOB be the diameter of the surface of the water before it is raised by the cylinder, then the cylinder GH (=the frustum A BCD) is to be a maximum.



As $1 : \frac{1}{2} :: 5^3 :: AB^3 = 1\frac{2}{8}\frac{5}{5}$, or $AB = \frac{5}{2} = 2a$.

Let now x=FG=the diameter of the base of the cylinder, then OE=b=3, and (by Vince's Flux, page 21) DC $=\frac{3}{4}x$, and by sim. tri. as $a : b :: \frac{3}{4}x : \frac{3b}{4a}$ $\therefore x$=IE $\therefore$ also FH=$\frac{1}{3}$IE=$\frac{bx}{4a}$; hence if p=.7854 the solidity of the cone CDE=$\frac{9px^2}{4} \times \frac{bx}{4a}$, and of the cone ABE=$\frac{4pa^2b}{3}$, therefore the difference of these, $\frac{9px^2}{4} \times \frac{bx}{4a} - \frac{4p^2a^2b}{3} = \frac{pbx^3}{4a}$=the solidity of the cylinder or by reduction $15x^3 = 64a^3$, whence $x = \frac{1}{3}\sqrt[3]{225} = 2.0274$, and FH =the axis of the cylinder=$\frac{1}{3}\sqrt[3]{225} = 1.2164$.

SECOND SOLUTION.—*By Mr. Farrand, Benedict, Montezuma.*

Let A=the contents of the cylinder and B=that part of the cone which the water at first occupies; then by the conditions of the question A+B=a max. and consequently since B is constant, A is a maximum, therefore (Vince's Flux. page 21) denoting the height of the cylinder by x, $3x$ will denote the height of the cone whose capacity is A+B. Put b=the radius of the base and h=the height of the given cone then $A = \frac{4\pi b^2}{h^2}x^3$, $B = \frac{\pi h b^2}{24}$, and moreover $A+B = \frac{9\pi h b^2}{h^2}x^3$, equating these two expressions for the value of A+B, and reducing, we have $x = \frac{h}{\sqrt[3]{120}}$ =1.216 nearly and hence the diam. of the cylinder base is 2.027

QUESTION XIII.—*By Patterson Wallace, New-York.*

Given the sum of the sides of a right angled plane triangle, and the difference between the hypothenuse and the perpendicular let fall on it from the right angle, to determine the sides of the triangle.

FIRST SOLUTION.—*By Mr. John Rochford, N. Y.*

Put x=perpendicular from the right angle on the hypothenuse, x=the hypothenuse above said perpendicular, then if b=the sum of the legs, it is well known that $(a+2x)^2=b^2+x^2$; from which $3x^2+4ax=b^2-a^2$; this divided by 3 and the square completed H. $x=-\frac{2}{3}a+\frac{(3b^2+a^2)^{\frac{1}{2}}}{3}$ from which the hypothenuse and legs are easily deduced.

SECOND SOLUTION.—*By Mr. Ingersol Bowditch, Boston.*

Let x and y be the sides, p the perpendicular, and z the hypothenuse of the triangle, a and b the given quantities, then per question $x+y=a$, or $y=a-x$, $z-p=b$, or $p=z-b$, $x^2+y^2=z^2$, or $x^2=z^2-y^2$, also $z:x::y:p$, or $xy=pz$, or $y=\frac{yz}{x}$. Then from the two values of y we get $ax-x^2=pz$: substituting in this the above value of p, we get $ax-x^2=z^2-zb$. Again, having as above, $x^2=z^2-y^2$, and $y=a-x$, we get $x^2=z^2-a^2+2ay-x^2$, or $ax-x^2=\frac{a^2-z^2}{2}$; equating their values of $ax-x^2$ we obtain $z^2-bz=\frac{a^2-z^2}{2}$, from which quadratic $z=\frac{\sqrt{3a^2+b^2}+b}{3}$, and $p=\frac{\sqrt{3a^2+b^2}-b}{3}$, and consequently making $pz=c$, $x=\frac{a+\sqrt{a^2-4c}}{2}$ and $y=\frac{a-\sqrt{a^2-4c}}{2}$.

THIRD SOLUTION.—*By Mr. B. McGowan, New-York.*

Let a= half sum and x= half difference of the required legs and y= the perpendicular, and therefore $y+b$= the hypothenuse, b being the given difference. Then $(a+a)^2+(a-a)^2=(y+b)^2$, or $2a^2+2x^2=x^2+2by+b^2$; also $(x+b).y=y^2+by=a^2-x^2$, and $2a^2-2x^2=2y^2+2by\therefore 4y^2+4by+b^2=4a^2$, and $y=-\frac{2b}{3}+\frac{1}{3}\sqrt{12a^2-b^2}$.

Mr. Solomon Wright, of Lumberville, Pa. solved this question by a similar method of notation to that of Mr. McGowan.

FOURTH SOLUTION.—*By Mr. C. Wilder, Baltimore.*

Put x for the hypothenuse of the triangle, a for the difference between the hypothenuse and perpendicular, and s for the sum of the legs, then the perpendicular will be $x-a$, and since the sum of the squares of the legs is equal to the square of the hypothenuse, and their rectangle equal to that of the hypothenuse and perpendicular, we shall have $x^2+2.x(x-a)=s^2$, or $3x^2-2ax=s^2$; and from this quadratic the hypothenuse x becomes known, and conse-

quently the perpendicular, and from these the legs are easily determined.

Mr. Phillips, of Harlem, remarks at the end of his geometrical solution, that this question of Mr. Wallace is only a particular case of prob. 62, Davis' Mathematical Companion for the year 1807. The problem alluded to is by Mr. Swale, a very ingenious geometrician of Liverpool, author of *Geometrical Amusements*, and editor of a periodical work entitled *Apollonius;* it is in these terms: Given the vertical angle, the sum or difference of the sides, and the sum or difference of the base and a straight line drawn from the vertex to make any given angle with the base, to construct the triangle.

QUESTION XIV.—*By Mr. Philip W. Hanson, New-York.*

Given the hypothenuse 60°, and the sum of the legs 70° of a right angled spherical triangle, to determine the legs of the triangle.

FIRST SOLUTION.—*By Mr. John Rochford, New-York.*

The formula for solviug this problem is given in Gregory's Trig. art. 30, page 236, which is there investigated.

Put a=the hyp. b and c the legs about the right angle—then per case 2. art 27 of that work 2 cos. a=2 cos. b. cos. c=cos. $(b=c)$+cos. $(b-c)$; hence cos. $(b-c)$=2 cos. a—cos. $b+c$. Whence $b-c$=48° 51′ nearly, and the legs are 59° 29′½, 10° 34′½.

SECOND SOLUTION.—*By Mr. B. M'Gowan, New-York.*

Let $x=\frac{1}{2}$ diff. of the legs, then $35°+x$, $35°-x$ are the sides, $\therefore$ cos. $(35°+x)+$cos. $(35°-x)$=cos. 60°; hence by developing and substituting, cos. ^{2}35°. cos. ^2x—sin. ^{2}35°+sin. ^{2}35°. cos. ^2x =cos. 60°, wherefore cos. ^2x=cos. 60°+sin. ^{2}35°, cos. $x=\sqrt{\frac{1}{2}+(.5735764)^2}$=.9104888, hence x=25° 25′ 37″, $35°+x$= 59° 25′ 37″=the greater leg, and $35°-x$=10° 34′ 23″.

THIRD SOLUTION.—*By Aliquis, Herkimer County, N. Y.*

Let x and y be the cosines of the two sides, then $xy=\frac{1}{2}$, and xy $-(1-x^2-y^2+x^2y^2)$=.34202, from which by eliminating x we have y^4-1 . $22504y^4$=. 26, whence y=.98302, x=.50833, and the two legs are 10° 34′ and 59° 26′.

FOURTH SOLUTION—*By Mr. Farquhar, Alexandria, D. C.*

Put the given hyp. (60°)=H, and sum of the legs (70°)—S, then prop. 20 Simpson's Trig. 2 cos. H—cos. S=cos. difference of the two legs: which difference being added to and substracted from the sum, the results divided by two will be the legs required.

QUESTION XV.—*By Eboracensis.*

To inscribe the greatest rectangle in one of the ovals of a lemniscate, of which the equation is $a^4y^2=a^4a^2-x^6$

FIRST SOLUTION.—*By Mr. John Rochford, New-York.*

Put $x=az^{\frac{1}{2}}$ then $y^2=a^2(z-z^3)=a^2(z'-z'^3)$ for the other abscissa, hence by substraction $a^2(z'-z)-a^2(z'^3-z^3)=0$, or $z'^2+z'z+z^2=1$. From this $z'=\frac{(4-3z^2)^{\frac{1}{2}}}{2}-z$, and the area of the triangle $(z'^{\frac{1}{2}}-z^{\frac{1}{2}})ay\left(\frac{\left(\frac{(4-3z^2)^{\frac{1}{2}}}{2}-z\right)^{\frac{1}{2}}-z^{\frac{1}{2}}}{2}\right)a^2(z-z^3)=$max. which differentiated will be $(4-18z^2+12z^4)\cdot\left((4z^2-3z^4)^{\frac{1}{2}}-z^2\right)+(2z-4z^3)\cdot(4-3z^2)^{\frac{1}{2}}\left((4z^2-3z^4)^2-z^2\right)^{\frac{1}{2}}-2\sqrt{2}(1-3z^2)(4-3z^2)^{\frac{1}{2}}\left(4z^2-3z^4\right)^{\frac{1}{2}}-z^2\Big)-2\sqrt{2}(2-3z^2)-z\sqrt{4-3z^2}=0$, which equation solved in its present state, by approximation or cleared from radicals and solved, will give the value of z, and then the values of x and y are easily had.

SECOND SOLUTION.—*By Dr. Bowditch, Boston.*

This question will be more simple if we suppose x to be expressed in terms of a taken for unity; and we shall have $x^6-x^2+y^2=0$, a cubic in x^2, one negative root of which may be neglected, as it produces imaginary values of x, and two positive roots (corresponding to equal values of of y) which we shall denote by x^2 and x'^2. These are obtained by the usual rules as in page 57 of Taylor's logarithms, or in page 262 of Barlow's tables, by putting $y^2\sqrt{\frac{27}{4}}=$ taking the square roots of these and retaining only the positive cos. z, and then $x^2=\sqrt{\frac{4}{3}}\cdot\cos.\left(60+\frac{z}{3}\right)$; $x'^2=\sqrt{\frac{4}{3}}\cdot\cos.\left(60-\frac{z}{3}\right)$ values of x and x', we get $x'-x=(\frac{4}{3})^{\frac{1}{4}}\left\{\sqrt{\cos.(60-\frac{z}{3})}-\sqrt{\cos.(60+\frac{z}{3})}\right\}$ and as $y=(\frac{4}{27})^{\frac{1}{4}}\sqrt{\cos.\ z}$ we get $y(x'-x)=\frac{2}{3}\sqrt{\cos.\ z}\left\{\sqrt{\cos.(60-\frac{z}{3})}-\sqrt{\cos.(60+\frac{z}{3})}\right\}$. This qnantity is per question to be a maximum, and its square also, or simply cos. $z\left\{\cos.(60-\frac{z}{3})+\cos.(60+\frac{z}{3})-2\sqrt{\cos.(60-\frac{z}{3})\cdot\cos.(60+\frac{z}{3})}\right\}$. If we now put $t=\cos.\frac{z}{3}$, we shall have $\cos.(60°-\frac{z}{3})=\frac{1}{2}t+\sqrt{\frac{3}{2}}\sqrt{1-t^2}$, $\cos.(60°+\frac{z}{3})=\frac{1}{2}t-\sqrt{\frac{3}{2}}\sqrt{1-t^2}$, $\cos.\ z=4.(\cos.\frac{z}{3})^3-3\cos.\frac{z}{3}=4t^3-3t$, and the product $\cos.(60°-\frac{z}{3}).\cos.(60°+\frac{z}{3})=\frac{1}{4}t^2-\frac{3}{4}(1-t^2)=\frac{4t^2-3}{4}$, consequently the last mentioned quantity which was to be a maximum becomes $\frac{4t^3-3t}{4}\left\{t-\sqrt{4t^2-3}\right\}$, taking its differential putting it=0, and redu-

cing it becomes $16t^3-6t=(16t^2-3).\sqrt{4t^2-3}$, squaring, reducing and multiplying by $\frac{2}{3}$, it becomes $512t^6-640t^4+192t^2-18=0$, putting $t^2=\frac{w}{8}+\frac{5}{12}$ and reducing it becomes $w^3-\frac{28}{3}w-\frac{326}{27}=0$, a cubic equation which reduced by the usual rules gives $w=3\,.\,566356\,.\,\therefore\ t^2=\frac{w}{8}+\frac{5}{12}=0.862421$, hence we get cos. $\frac{z}{3}=t$, and $\frac{z}{3}=21°\,.\,46'\,7''\,.\,7$, $z=65°\,18'\,23''\,.\,1$, with these values we get $x=0\,.\,4065906$, $x'=0\,.\,9523867$, $7=0\,.\,4009960$

QUESTION XVI.—*By Mr. A. B. Quinby, New-York.*

If an infinitely small straight rod, that is suspended at one of its extremities, be raised till it form any given angle with the vertical, and then allowed to descend, it is required to determine the position it will have when the quantity of its motion that it will have in a vertical direction, is a maximum.

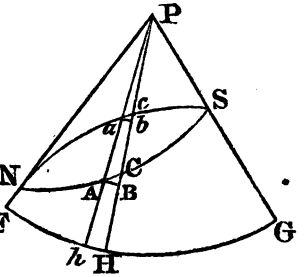
SOLUTION.—*By the Proposer, Mr. Quinby, N. York.*

Let $a=$ the sine of the given angle that the rod makes with the horizontal line passing through the point of suspension, and $x=$ the sine of the angle that the rod makes with the same horizontal line when its motion in a vertical direction is a maximum—then $\sqrt{2g}\times\sqrt{x\pm a}\times\sqrt{r^2-x^2}=$ max. or $(x\pm a)\,.\,(r^2-x^2)=$max. And $x\,(r^2-x^2)-2ax\,(x\pm r)=0$, or $3x^2\pm 2ax=r^2$, whence $x=\frac{\sqrt{3r^2+a^2}\pm a}{3}$.

QUESTION XVII.—*By Mr. John Rochford, New-York.*

Two ships set sail at the same instant from Sligo for New-York, one in the arc of a great circle, the other on the rhumb line, connecting both cities, and keep always due north and south of each other, required their latitudes and longitudes when they are the greatest possible distance asunder.

SOLUTION—*By Dr. Bowditch, Boston.*



Let P be the pole, S Sligo, N New-York, F h H G the equator, N $a\,c$ S a great circle, and NACS a rhumb line passing through N S. Draw the meridian P S G, P N F, also P $c\,b$ H, P a A h, including the small angle of longitude h P H. Let L=the latitude of Sligo, L'= the latitude of New-York, D= their difference of longitude FPG, C=the course from Sligo to New-York on the rhumb line counted from the south=the angle A C B. Let l and l' be the latitudes of the ships at C and c, and G P H= the longitude counted from Sligo=δ, and H P $h=d\delta$. Draw ab,

A B parallel to the equator. Then the difference of latitude of the two ships on P H is C c, and on P h is A a, and the ships will be at the greatest distance asunder when these distances are equal, or their differential is nothing; consequently the question is satisfied by putting A a=C c; but by construction A a=B b, consequently B b=C c, and therefore B C=bc. Now A B=$d\delta$ cos. l. B C= A B cot. c, = $d\delta$ cos. l. cot. c; and for the point of maximum this gives $bc=d\delta$ cos. l, cot. c: but $ab=d\delta$ co. l', hence $\frac{ab}{bc}$ = tan. bca=tan. P cS=$\frac{d\delta \cos.\ l'.}{d\delta \cos.\ l.\ \cot.\ c.}=\frac{\cos.\ l'}{\cos.\ l.\ \cot.\ c}$, so that the question is reduced to this, to find on the great circle N a c S the point c, such that tan. P c S=$\frac{\cos.\ l'}{\cos.\ l}$ tan. c. On the rhumb line, as above, we have $-d\,l.=d\delta$ cos. l. cot. c, $\therefore\ d\delta \cot.\ c=-\frac{d\,l.}{\cos.\ l.}$ whose integral is δ cot. c=hyp. log. $\left\{\frac{\text{tan. } 45°+\frac{1}{2}\text{ L,}}{\text{tan. } 45°+\frac{1}{2}\ l}\right\}$ or by putting a= 2.302585; $\frac{\delta \cot.\ c}{a}$= log. tan. $(45+\frac{1}{2}$ L) — log. tan. $(45+\frac{1}{2}\ l)$ using common tabular logarithms, so that to find the points c, C, we have these equations

$$\text{log. tang. } (45+\tfrac{1}{2}\ l) = \text{log. tang. } (45'+\tfrac{1}{2}\text{ L}) - \frac{\delta \cot.\ c}{a},$$

$$\text{tang. P } c\text{ S} = \frac{\cos.\ l'}{\cos.\ l}\ \text{tang. } c;$$

from which, by a few trials the values of δ, l, l', may be found.

QUESTION XVIII. OR PRIZE QUESTION—*By Professor Adrain, Columbia College, New-York.*

It is required to investigate the path that ought to be described by a boat in crossing a river of given breadth from a given point on one side, to a given point on the other, so as to make the passage in the least time possible; supposing the simple velocity of the boat by the propelling power to be given; and that the velocity of the current, being in the same direction with the parallel sides of the river, is variable and expressed by any given function of the perpendicular distance from that side of the river from which the boat sets out.

PRIZE SOLUTION—*By Professor Strong, Hamilton College, Oneida County, State of New-York.*

It is manifest that the boat, by the propelling power alone, will describe a certain line, either straight or curved, passing from her point of departure to the other side of the river, which is such that the current will float her down the river into another curve, which is formed by the composition of the velocity of the boat in the direction of the first curve and of the velocity of the current, and

that the curve thus described, from the point of departure to the given point of arrival, will be described in the same time that the propelling power alone would cause her to describe the first curve mentioned, which time, by the question, is to be a minimum.

Let then $y \,..\, y'$ denote corresponding ordinates of the two curves, (y belonging to the first curve,) having x for their common abscissa, the origin of the co-ordinates being at the point of departure, the perpendicular width of the river being the line of the abscissas, and its side the line of the ordinates.

Let V denote the given velocity of the propelling power, and t the time elapsed from the instant of departure; also let φx vary as the velocity of the current, φx denoting some function of x, then may the velocity of the current at the distance x be denoted by $a \,.\, \varphi x$, a being a constant quantity, and for simplicity put $a\varphi x = x'$.

Now we have $\sqrt{ax^2+dy^2} = \mathrm{V}dt$, $\because dy^2 = \mathrm{V}^2dt^2 - dx^2$, also $dy - dy = x'dt$, for $dy' - dy$ manifestly denotes the infinitely small space through which the current floats the boat in the time dt, $\therefore dy^2 = (x'dt - dy')^2 = x'^2dt^2 - 2x'dtdy' + dy'^2 = \mathrm{V}^2dt^2 - dx^2$, (since $dy^2 = \mathrm{V}^2dt^2 - dx^2$,) hence by transformation we have $(\mathrm{V}^2 - x'^2)\,dt^2 + 2x'dy'dt = dx^2 + dy'^2$, or we have $dt^2 + \frac{2x'dy'}{\mathrm{V}^2 - x'^2}\,dt = \frac{dx^2 + dy'^2}{\mathrm{V}^2 - x'^2}$, $\because$ complete the square, extract the root, and transpose, and we have

$$dt = \frac{\sqrt{\mathrm{V}^2 \,.\, dy'^2 + (\mathrm{V}^2 - x'^2)dx^2} - x'dy}{\mathrm{V}^2 - x'^2}.$$

Put $\frac{dy'}{dx} = p$ and when $dt = \frac{\sqrt{\mathrm{V}^2p^2 + \mathrm{V}^2 - x'^2} - x'p}{\mathrm{V}^2 - x'^2}$,

$$\because t = \int dx \left(\frac{\sqrt{\mathrm{V}^2p^2 + \mathrm{V}^2 - x'^2} - x'p}{\mathrm{V}^2 - x'^2} \right).$$

This integral is to be taken from the given point of departure to the given point of arrival.

But t is to be a minimum, $\because$ its equal

$$\int dx \left(\frac{\sqrt{\mathrm{V}^2p^2 + \mathrm{V}^2 - x'^2} - x'p}{\mathrm{V}^2 - x'^2} \right) \text{ must be a minimum also.}$$

Hence by the method of variations regarding dx and x' (which is a function of x alone), as constant we must have

$$\delta \int dx \left(\frac{\sqrt{\mathrm{V}^2p^2 + \mathrm{V}^2 - x'^2} - x'p}{\mathrm{V}^2 - x'^2} \right) =$$

$$\int \frac{dx\,\delta p}{\mathrm{V}^2 - x'^2} \left(\frac{\mathrm{V}^2p}{\sqrt{\mathrm{V}^2p^2 + \mathrm{V}^2 - x'^2}} - x' \right) =$$

$$\int \frac{d\,\delta y'}{\mathrm{V}^2 - x'^2} \left(\frac{\mathrm{V}^2p}{\sqrt{\mathrm{V}^2p^2 + \mathrm{V}^2 - x'^2}} - x' \right)$$

(by putting par δp its value $\frac{\delta dy'}{dx}$) =

$$\varphi'\delta y'''-\phi\delta y'\int\delta y\,.\,d\left(\frac{1}{V^2-x'^2}\left(\frac{V^2p}{\sqrt{V^2p^2-V^2x'^2}}x'\right)\right)=0,$$

($\delta y'''$ being the value of δy at the point of arrival, and $\delta y''$ its value at the point of departure, φ' is the value of $\frac{1}{V^2-x'^2}\left(\frac{V^2p}{\sqrt{V^2p^2+V^2-x'^2}}x'\right)$ at the point of arrival, and ϕ the the value of the same quantity at the point of departure).

But the extremities of the curve being given points $\cdots\delta y'''=o$, and $\delta y''=o$, hence we have

$$\int\delta y'.d\left(\frac{1}{V^2-x'^2}\left(\frac{V^2p}{\sqrt{V^2p^2+V^2-30'^2}}x'\right)\right)=o$$

but $\delta y'$ is indefinite or represents the variation of y at any point of the curve, and therefore cannot be universally$=o$, $\cdots$ the quantity by which $\delta y'$ is multiplied must$=o$, hence

$d\left(\frac{1}{V^2-x'^2}\left(\frac{V^2p}{\sqrt{V^2p^2+V^2-a'^2}}-x'\right)\right)=o$ therefore by integration

$\frac{1}{V^2-x'^2}\left(\frac{V^2p}{\sqrt{V^2p^2+V^2-x'^2}}\right)=\frac{1}{c}$—whence by reduction

$$p=\frac{V^2-x'^2+cx'}{\sqrt{(c-x')^2-V^2}},\text{ or }\frac{dy}{dx}=\frac{V^2-x'^2+cx'}{\sqrt{c-x'^2-V^2}}$$

which is the differential equation of the required curve, which the boat actually describes by compounding the velocity of the propelling power with that of the current.

Also from the equation $dt=dx\left(\frac{\sqrt{V^2p^2+V^2-x'^2}-x'p}{V^2-x'^2}\right)$, I derive $dt=dx\,.\,\frac{cV^2p-x'(V^2-x'^2+cx')p}{(V^2-x^2-x'^2+cx)}$ by substituting for $\sqrt{V^2p^2+V^2-x'^2}$ its value $\frac{cV^2p}{V^2-x'^2+cx}$, or by reduction $dt=\frac{dx\,.\,p(c-x')}{V^2-x'^2+cx'}=\frac{dx\,.\,(c-x')}{-V\sqrt{(c-x)^2-V^2}}$, by substituting for p its value $\frac{V^2-x'^2+cx'}{V\sqrt{(c-x')^2-V^2}}$: this equation being integrated gives us the time.

Also from this equation we have $Vdt=\sqrt{dx^2+dy}$ $=\frac{dx(c-x')}{\sqrt{(c-x')^2-V^2}}$, or $1+\frac{dy^2}{dx^2}=\frac{(c-x')^2}{(c-x')^2-V^2}\therefore\frac{dy}{dx}=\frac{V}{\sqrt{(c-x')^2-V^2}}$ for the differential equation of the curve which the boat ought to describe by the propelling power alone; hence if the boat is propelled continually in the direction of the corresponding points of this curve, the current will float her down into the other curve, and she will make the passage in the shortest time possible.

From the equations $\frac{dy}{dx}=\frac{V^2-x'^2+cx'}{V\sqrt{(c-x')^2-V^2}}$, and $\frac{dy}{dx}=$

$\frac{V}{\sqrt{(c-x')^2-V^2}}$, it is manifest that if x' is formed of terms consisting only of direct powers of x multiplied by constant co-efficients, when $x=o$, or at the beginning of the motion, we shall have $x=o$, $\therefore \frac{dy'}{dx}=\frac{V}{\sqrt{c^2-V^2}}$, and $\frac{dy}{dx}=\frac{V}{\sqrt{c^2-V^2}}$, $\therefore \frac{dy}{dx}=\frac{dy}{dx}$ in this case, therefore the two curves touch each other at the origin of the motion. But if x' should have some terms which are not multiplied either by x or its powers, then will the two curves cut each other at a given angle at the origin But if x' should consist partly of given quantities without x, and of direct and inverse powers of x, or simply of inverse powers of x, then when $x=o$, we shall have $\frac{dy'}{dx}=$infinity, and $\frac{dy}{dx}=\frac{V}{\text{infinity}}$, $\therefore \frac{dy}{dx}=o$ in this case, and $\frac{dy}{dx}=$ infinity $\therefore$ the two curves cut each other at right angles, and the boat must commence her motion in a direction which is at right angles to the direction of the current.

With respect to the determination of the constant c, it may be found in the following manner. The above equations may be exhibited as follows, $dy'=\frac{dy\,(V^2-x'^2+cx')}{V\sqrt{(c-x')^2-V^2}}$ and $dy=\frac{dx\,V}{\sqrt{(c-x')^2-V^2}}$; then the composition of x' in terms of x being known, we find y' and y in terms of x and constants (by integration), and, by correctious, the constants which these integrations introduce will be determined in terms of c, by considering that y', y, x are each nothing at the origin of the motion $\therefore y'y$, in terms of x, c and given quantities: put then in the equation in y' its value at the given point of arrival, and for x the breadth of the river, and we shall have an equation involving c and known quantities. This equation being solved with respect to c, will give c in known quantities, which beino found then y' and y will be exhibited in terms and known quantities, by means of which the two curves may be constructed by the ordinary methods. But if c thus found should give for y' and y imaginary values, it will show that the boat cannot be directed so as to reach the point desired with the given velocity V. Also, c being determined, then may the time t be found by the integration of the equation $dt=\frac{dx\,(c-x')}{\sqrt{(c-x')^2-V^2}}$.

It should be noted that if the point of arrival is higher up the stream than the axis of x, the value of y' corresponding must be considered as negative in the determination of c, and should any values of y' come out negative, then the corresponding points of the curve lie above the axis of a.

I intended to have set down several species of these curves, but I find that I shall not have sufficient room for it: I shall, therefore, only consider the simplest, which is that which arises by supposing $x'=$ the velocity of the current to be constant. In this case the

equations $\frac{dy'}{dx} = \frac{V^2 - x'^2 + cx'}{V\sqrt{(c-x')^2 - V^2}}$, $\frac{dy}{dx} = \frac{V}{\sqrt{(c-x')^2 - V^2}}$ show that the curves become right lines, which make angles with the axis, whose tangents are $\frac{V^2 - x'^2 + cx'}{V\sqrt{(c-x')^2 - V^2}}$, $\frac{V}{\sqrt{(c-x')^2 - V^2}}$, $\because dy' : dy :: V^2 - x'^2 + cx' : V^2 \because dy' - dy : dy :: (c-x')\,x' : V^2$, but $dy' - dy : dy :: x' : v \because x' : v :: (c-x')\,x' \cdot V^2$, $\because (c-x') = \frac{V^2}{v}$, ($v$ being that part of V which carries the boat in the direction of the current; hence $\frac{dx'}{dx} = \frac{v + x'}{\sqrt{V^2 - v^2}}$, $\frac{dy}{dx} = \frac{v}{\sqrt{V^2 - v^2}}$, as they manifestly ought to be when the point of arrival is below the line of the abscissas. But when it lies above, then x' is to be considered as negative, and the equations resulting are

$$\frac{dx'}{dx} = \frac{v - x'}{\sqrt{V^2 - v^2}} \text{ and } \frac{dy}{dx} = \frac{v}{\sqrt{V^2 - v^2}}.$$

ACKNOWLEDGMENTS, &c.

The following ingenious gentlemen favoured the Editor with solutions to the questions in No. 2, Article V. The figures annexed to the names refer to the questions answered by each as numbered in that article.

Dr. Bowditch, of Boston, and Professor Strong, Hamilton College, each most ingeniously answered all the questions.

John Rochford, N. Y. and Aliquis, Herkimer Co. N. Y. each answered all the questions but the 18th. Charles Farquhar, Alexandria, D. C and Charles Wilder, Baltimore, each answered all all but the 17th and 18th. Walter Nichols, N. Y. all but the 15th, 17th, 18th. B. Mc Gowan, N. Y. all but the 12th, 17th, 18th. James Phillips, Harlem, and Farrel Ward, N. Y. each answered all but 15, 16, 17, 18. Rev. Dr. Clowes, Chestertown, Md. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14. J. Ingersoll Bowditch, Boston, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14. Farrand Benedict, Montezuma, N. Y. 1, 2, 3, 4, 6, 7, 8, 11, 12, 13, 14, 16. Rev. Henry Doyle, N. Y. 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14. Cornelius Davis, Dutchess Co. N. Y. 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13. Joseph Strode, Phil. 1, 2, 3, 4, 6, 7, 8, 10, 11, 13. Wm. Lenhart, Phil. 1, 2, 3, 4, 5, 6, 7, 8, 9. James Hamilton, Trenton, N. J. 1, 2, 3, 6, 8, 9, 11. Cyril Paschalis, and Wm. H. Sidell, N. Y. each 1, 2, 3, 6, 11, 14. Q. U. Z. Oxford, Ohio, 1, 2, 4, 6, 11, 13. Henry Darnall, Phil. 1, 2, 3, 6, 11, 13. James Quin, N. Y. 1, 2, 3, 4, 6, 10. Michael Floy, N. Y. 1, 2, 3, 6, 7, 8. Daniel D. Aikin, Dutchess Co. N. Y. 1, 2, 11, 13. Nicholas Donnelly, N. Y. 1, 2, 6, 8, 9. David S. Bogart, N. Y. 1, 2, 3, 4. Wm. Scally, N. Y.

1, 2, 6, 11. John Capp, Esq, Harrisburg, 6, 7, 8, 9. Daniel Shanley, Charleston, S. C. 1, 3, 6, 11. James Warnock, N. Y. 1, 2, 3, 7. Wm. Kells, Bergen, N. J. 1, 6, 11, 15. Charles Potts, Phil. 6, 11, 13. Dennis Keenan, N. Y. 1, 2, 11. Samuel Bard McVichar, N. Y. 1, 6. Wm. Vogdes Edgemont, Del. Co. Penn. 1. James McGinnis, Harrisburg, 11. Solomon Wright, Lumberville, Bucks Co. Penn. 13. A. B. Quinby, N. Y. 16. Denis Carmody, N. Y. 7. Peter Fleming, N. Y. 4. Tyro, Lexington, Ky. 12. Edward Ward, N. Y. 11. Dr. Henry J. Anderson, N. Y. 18, or Prize Question.

The Prize has been awarded to Professor Strong, Hamilton College, State of New-York.

Of the solutions to the prize question by Dr. Bowditch and Dr. Anderson, both were effected by the method of variations, and terminated in the same general equation of the required curve.

The communications of several correspondents were received too late to be made use of in publishing solutions to the questions proposed in No. II.

ARTICLE VII.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN N°. IV.

QUESTION I. (39.)—*By Mr. D. Shanley, Charleston, S. C.*

Given the two equations,

$$9\frac{x^2}{y^2}+36\frac{x}{y}=85,$$

$$3\frac{xy^3}{5}+\frac{51\,xy^2}{10}=\frac{102\,x}{5}+16;$$

to find the values of x and y.

QUESTION II. (40.)—*By Mr. Michael Floy, New-York.*

Required two such numbers that their difference may be equal to 4, and the product of their product, and the sum of their squares, may be 480.

QUESTION III. (41.)—*By the same.*

Given the sum of the cubes of two numbers=35 and the sum of their ninth powers=20195 to find the numbers.

QUESTION IV. (42.)—*By Mr. William Lenhart, Philadelphia.*

Given the three following equations,

$$\frac{x}{y}\times\frac{1}{x^4}=a, \quad \frac{z}{x}\times\frac{1}{y^4}=b, \quad \frac{y}{z}\times\frac{1}{x^4}=c;$$

to determine the values of x, y, z, in terms of the given quantities a, b, c.

QUESTION V. (43.)—*By Mr. Farrell Ward, New-York.*

To Find a rational right-angled triangle, such that, if the digits in each side separately be added together, the three sums shall be rational squares.

QUESTION VI. (44.)—*By Mr. John Rochford, New-York.*

Find rational right-angled triangles of equal perimeters, so that the area of each will be m times a square number, and exhibit the sides of three of them in integers, such that the sum of their squares will be equal to a square.

QUESTION VII. (45.)—*By Mr. Charles Potts, Philadelphia.*

The sides of a triangle are in arithmetical progression of which the common difference is given $= a$, and the sum of the cubes of the sides is given $= b$; to determine the sides of the triangle.

QUESTION VIII. (46.)—*By John Capp, Esq. Harrisburg, Pa.*

A manufacturer at the Salt-works in the state of New-York, during the late war, had on hand a thousand barrels of salt, valued at two dollars per barrel at the works, engaged a carrier who agreed to carry salt at the rate of five dollars per barrel to Pittsburg, and to take one moiety of his wages in salt at the works, and the other moiety at Pittsburg on the delivery of the residue: how much salt did the carrier receive at each place? and what was the gain or loss of the manufacturer in the transaction, supposing salt to have been rated at ten dollars per barrel at Pittsburg?

QUESTION IX. (47.)—*By Mr. William Kells, Bergen, N. J.*

It is required to find the diameter of a vessel in the form of a hemispherical surface, which would cost as much to buy it full of wine worth a dollar per gallon as would buy a lid for the same worth six dollars per square inch.

QUESTION X. (42.)—*By Mr. S. Wright, Bucks Co. Pa.*

Required the dimensions of a right-angled triangle, the hypothenuse of which is given, the perpendicular added to twice the base being a maximum.

QUESTION XI. (43.)—*By Mr. Henry Darnell, Philadelpha.*

A ship came to anchor, and after paying out 40 fathoms of cable she was fifty fathoms from her buoy, which floated perpendicularly over her anchor: the depth of the water is required.

QUESTION XII. (44.)—*By Mr. James Phillips, Harlem.*

Three masses are placed by pairs at the extremities of an uniform rigid bar void of gravity, whose length is 2040 inches, in such a manner that if G be the common centre of gravity of A and B, and B be removed and C put in its place, the distance of the common centre of gravity of A and C from G will be 675 inches nearer the extremity B; but if B be now placed with A, then the distance of the common centre of gravity of A+B and C from that of A and C is 216 inches nearer the extremity A. Now if the sum of these be 20 lbs. what is the weight of each and the distance AG?

QUESTION XIII. (45.)—*By Professor Strong, Hamilton College, State of N. York.*

Integrate the differential equation

$$dy = \frac{dx}{(1-x^3)^{\frac{1}{2}}}$$

in finite terms if it be possible.

QUESTION XIV. (46.)—*By Eboracensis.*

To inscribe the greatest rectangle in one of the ovals of a lemniscate, of which the equation is $a^6y^2=a^6x^2-x^8$.

QUESTION XV. (47.)—**OR PRIZE QUESTION.**

By Eboracensis.

On one of the extensive plains in the state of Illinois there rises a hill of great beauty and perfect uniformity of curvature. At four different points on the plane of its base, at different distances from it the angles of elevation of the hill were accurately determined by observation, and the mutual distances of the places of observation were ascertained by the most approved methods of admeasurement and calculation; with these data, the determination of the perpendicular altitude of the hill above the plane of its base is respectfully submitted to mathematicians for their investigation.

THE

MATHEMATICAL DIARY,

N° III.

BEING THE PRIZE NUMBER OF DR. HENRY J. ANDERSON, NEW-YORK.

ARTICLE VIII.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE VII. NO. III.

QUESTION I.—*By Mr. D. Shanley, Charleston, S. C.*

Given the two equations,

$$9\frac{x^2}{y^2}+36\frac{x}{y}=85,$$

$$\frac{3xy^3}{5}+\frac{51xy^2}{10}=\frac{102x}{5}+16;$$

to find the values of x and y.

FIRST SOLUTION.—*By Patrick Byrne, Philadelphia.*

By adding 36 to both sides of the first given equation, and extracting the square root we get $3\frac{x}{y}+6=11$, whence $3x=5y$; and substituting this value for $3x$ in the second equation, we have

$$2y^4+17y^3-68y=32.$$

The roots of this equation, or the values of y, are 2, -8, -2, $-\frac{1}{2}$, and therefore $x=\frac{5y}{3}$ has the corresponding values

$$\frac{10}{3},\ -\frac{40}{3},\ -\frac{10}{3},\ -\frac{5}{6}.$$

SECOND SOLUTION.—*By S. of Brooklyn, L. I.*

From the first equation by quadratics we get $\frac{x}{y}=\frac{2}{3}$, or $x=\frac{5}{3}y$, which substituted in the second gives

$$10y^4+85y^3-340y=160,$$

from which we get $y=2$, and $x=\frac{5y}{3}=\frac{10}{3}$ as required.

THIRD SOLUTION.—*By Frederick Nott, Harlem.*

The first equation divided by 9 becomes

$$\frac{x^2}{y^2}+4\frac{x}{y}=\frac{85}{9},$$

hence by evolution we obtain $x=\frac{5y}{3}$. Now substitute this value of x in the second equation and we have

$$y^4+\frac{51}{6}y^3=34y+16,$$

from which the value of y can be found $=2$, and thence $x=\frac{10}{3}$.

QUESTION II.—*By Mr. Michael Floy, New-York.*

Required two such numbers that their difference may be equal to 4, and the product of their product, and the sum of their squares, may be 480.

FIRST SOLUTION.—*By Michael Doyle, Washington College, Chestertown, Maryland.*

Let $x+4$ and x be the numbers. Then by the question

$$(x^2+4x)\ (2x^2+8x+16)=480,$$

that is $2x^4+16x^3+48x^2+64x=480$,

and dividing by 2, $x^4+9x^3+24x^2+32x=240$,

whence $x=2$ and $4+2=6$; consequently 2 and 6 are the numbers.

SECOND SOLUTION.—*By Mr. —— Kean, New-York.*

Let the greater $=x+2$, the less $=x-2$, then their product $=(x+2).(x-2)=x^2-4$, and the sum of their squares $=(x+2)^2+(x-2)^2=2x^2+8$; therefore $(2x^2+8).(x^2-4)=2x^4-32=480$ by the question, hence $x=4$; consequently the greater $=4+2=6$ and the less $=4-2=2$.

THIRD SOLUTION.—*By John F. James, Trenton.*

Let $x=$ the less and $x+4=$ the greater, then $(x^2+4x)\times[(x+4)^2+x^2]=2x^4+16x^3+48x^2+64x=480$ by the question. Dividing by 2 and adding 16 to both sides of the equation we have $x^4+8x^3+24x^2+32x+16=256$; and extracting the fourth root we have $x+2=4$ whence $x=2$, and the numbers are 2 and 6.

FOURTH SOLUTION.—*By Jesse Willets, Maiden-Creek, Berk's County, Pennsylvania.*

Put $x=\frac{1}{2}$ sum and $a=\frac{1}{2}$ difference, then will $x+a=$ greater and $x-a=$ less; also put $b=480$, and by the question we have $2x^4-2a^4=b$, whence $x^4-a^4=\frac{b}{2}$, $x^4=a^4+\frac{b}{2}$ and $x+\sqrt[4]{(a^4+\frac{b}{2})}=4$, and $x+a=6$, $x-a=2$, the numbers sought.

FIFTH SOLUTION.—*By John Rochford, N. Y.*

Let $x=$ the sum of the squares, and $y=$ the product of the required numbers, then $xy=480$, and $x-2y=16=$ the square of the difference, as is well known. From the second $x=2y+16$, and this multiplied by y gives $xy=2y^2+16y=480$, or dividing by 2, $y^2+8y=240$, the square being completed and then solved, $y=12$, hence $x=40$. Now there are given the sum of the squares and product of two numbers to find them, which is a very familiar problem, and is done thus $(40\pm 2\times 12)^{\frac{1}{2}}=8$ and 4, the sum and difference, and then $\frac{8\pm 4}{2}=6$ and 2, the required numbers.

QUESTION III.—*By Mr. Michael Floy, New-York.*

Given the sum of the cubes of two numbers=35 and the sum of their ninth powers=20195 to find the numbers.

FIRST SOLUTION.—*By E. H. Rockwell, Frederick, Md.*

Put $a=35$, $b=20195$, and let $x=$ the cube of the less number then $a-x=$ cube of the greater, consequently $x^3=$ the 9th power of the greater, and $(a-x)^3=$ the 9th power of the less, and by the question we have $(a-x)^3+x^3=b$ which by involution becomes

$$a^3-3a^2x+3ax^2-x^3+x^3=b,$$

or $$3ax^2-3a^2x=b-a^3,$$

whence $$x^2-ax=\frac{b-a^3}{3a};$$

by completing the square we have

$$x^2-ax+\frac{a^2}{4}=\frac{b-a^3}{3a}+\frac{a^2}{4},$$

and by evolution $$x-\frac{a}{2}=\sqrt{\frac{b-a^3}{3a}+\frac{a^2}{4}}.$$

which in numbers is $x-17\frac{1}{2}=9\frac{1}{2}$, whence $x=27$, and $35-27=8$, therefore $\sqrt[3]{27}$ and $\sqrt[3]{8}$, that is, 3 and 2, are the numbers required.

SECOND SOLUTION.—*By John F. James, Trenton.*

Let x and y be the numbers, then per quest. $x^3+y^3=35$, and $x^9+y^9=20195$. From these we have $x^3=35-y^3$ and $x^9=20195-y^9$, the latter of which being the cube of the former we have $(35-y^3)^3=20195-y^9$,
which is reduceable to $105y^6-3675y^3=-22680$;
and this divided by 105, and resolved by completing the square, extracting the root, &c. gives $y=3$, and the numbers sought are 2 and 3.

THIRD SOLUTION.—*By Daniel Shanley, Charleston, S. C.*

Let x and y represent the two numbers, then

$$\left.\begin{array}{l} x^3\times y^3=35 \\ x^9+y^9=20195 \end{array}\right\} \text{ by the question.}$$

Dividing the second equation by the first we have

$$x^6-x^3y^3+y^6=577,$$

the first square is $x^6+2x^3y^3+y^6=1225$
by subtracting $3x^3y^3=648$ or $x^3y^3=216$.
Hence $x^6-2x^3y^3+y^6=361$
and taking the root $x^3-y^3=19$.
But $x^3+y^3=35$,
and by addition $2x^3=54$ or $x^3=27$, and $x=3$
Also by subtraction $2y^3=16$, or $y^3=8$, whence $y=2$, therefore 3 and 2 are the numbers.

FOURTH SOLUTION.—*By Benjamin Hallowell, Alexandria, D.C.*

Let $x^{\frac{1}{3}}$ and $y^{\frac{1}{3}}$ be the numbers, then $x+y=35$, and $x^3+y^3=20195$: the latter divided by the former gives $x^2-xy+y^2=577$, which subtracted from the square of the first leaves $3xy=648$, or $xy=216$: subtract therefrom the third equation, and we get $x^2-2xy+y^2=361$, or $x-y=19$, this added to and subtracted from the first gives $x=27$, $y=8$, hence the numbers are 3 and 2.

FIFTH SOLUTION.—*By Barclay Waterman, Philadelphia.*

Put x and y for the numbers sought, then per question

$$x^3+y^3=35, \text{ and } x^9+y^9=20195,$$

subtract the second equation from the cube of the first, and there remains $3x^3y^3\times(x^3+y^3)=22680$, or $3x^3y^3\times 35=22680$; from this $x=\frac{6}{y}$, put this for x in the first equation, and we get $y^6-35y^3=-216$, whence $y=3$, and $x=2$.

QUESTION IV.—*By Mr. William Lenhart, Philadelphia.*

Given the three following equations,

$$\frac{x}{y}\times\frac{1}{x^4}=a, \quad \frac{z}{x}\times\frac{1}{y^4}=b, \quad \frac{y}{z}\times\frac{1}{x^4}=c;$$

to determine the values of x, y, z, in terms of the given quantities a, b, c.

FIRST SOLUTION.—*By Henry Darnall, Philadelphia, and Michael Floy, New-York.*

Multiplying the three equations together we have $(xyz)^3=\frac{1}{abc}$ or $xyz=\sqrt[3]{\frac{1}{abc}}=p$, $\therefore x=\frac{p}{yz}$, but $x=ayz^4$, and $x=\frac{z}{by^4}$ in the 2d; whence $ayz^4=\frac{p}{yz}$ or $y^2=\frac{p}{az^5}$, also $\frac{z}{by^4}=\frac{p}{yz}$ from which $y^3=\frac{z^2}{pb}=\frac{p}{az^5}\times\sqrt{\frac{p}{az^5}}$, or $\frac{z^2}{pb}=\sqrt{\frac{p^3}{a^3z^{15}}}$, or $z=\left(\frac{b^2p^5}{a^3}\right)^{\frac{1}{19}}$, from which the values of x and y are easily found.

SECOND SOLUTION.—*By James Hamilton, Trenton.*

The given equations when cleared of fractions are $x=ayz$, $z=bxy^4$, $y=czx^4$.

Let $x=my$, $z=ny$, and by substitution we have $my=an^4y^5$, $ny=bmy^5$, $y=cnm^4y^5$.

The first of these three divided by second gives $\frac{m}{n}=\frac{an^4}{mb}$, whence $m^2=\frac{an^5}{b}$, or $m=\sqrt{\frac{an^5}{b}}$. Again the second divided by the third gives $n=\frac{bm}{cnm^4}=\frac{b}{c}\cdot\frac{1}{nm^3}$, or $m^3=\frac{b}{cn^2}$, therefore $m=\sqrt[3]{\frac{b}{cn^2}}$. Now equating the values m we have $\sqrt{\frac{an^5}{b}}=\sqrt[3]{\frac{b}{cn^2}}$, which by raising each side to the 6th power gives $\frac{a^3n^{15}}{b^3}=\frac{b^2}{c^2n^6}$ whence $n=\left(\frac{b^5}{a^3c^2}\right)^{\frac{1}{19}}$; and hence the values of x, y, z, are easily found.

THIRD SOLUTION.—*By Cornelius Davis, Dutchess Co. N. Y.*

Put $z=vx$, $y=wx$, and the equations become $vw^{-4}x^{-4}=b$, $wv^{-1}x^{-4}=c$, $w^{-1}v^{-4}x^{-4}=a$.

Dividing the first by the second, and the third by the first, we have $w^{-5}v^{2}=bc^{-1}$, $w^{-3}v^{5}=ab^{-1}$:

Now raise these equations to the third and fifth powers respectively, and we have $w^{15}v^{6}=b^{3}c^{-4}$, $w^{-15}v^{25}=a^{5}b^{-5}$,

the latter of which divided by the former gives $v^{19} = a^{5} b^{3} c^{3}$, from which v is known, and thence the quantities required.

FOURTH SOLUTION.—*By Martin Roche, Philadelphia.*

By multiplying the three given equations together we get $\frac{1}{x^4y^4z^4} = abc$ whence $xyz = \sqrt[4]{\frac{1}{abc}}$, and the solution may be similar to that given in the third No. of the Diary to Question 4, by Mr. Farquhar. Or by multiplying the given equations in pairs, &c. we get $x^5y^3 = \frac{1}{cb}$, $x^3z^5 = \frac{1}{ac}$, $y^5z^3 = \frac{1}{ab}$, whence the solution may be similar to that given to the same by Mr. Lenhart. Hence it is evident that this fourth question, as proposed by Mr. Lenhart, is only a transformation of the 4th question in the second No. proposed by Mr. Fleming.

QUESTION V.—*By Mr. Farrell Ward, New-York.*

To Find a rational right-angled triangle, such that, if the digits in each side separately be added together, the three sums shall be rational squares.

FIRST SOLUTION.—*By William Forrest, New-York.*

Assume $x^2 - y^2$, $2xy$, and $x^2 + y^2$ as the base perpendicular and hypothenuse of the required triangle. If we now take $x = 2$, $y = 1$, we have 3, 4, 5, for the three sides, and each of these numbers multiplied by 9 give 27, 36, and 45, which numbers evidently answer the conditions of the problem. If we assume other values for x and y we can obtain other answers to an unlimited extent.

SECOND SOLUTION.—*By John Capp, Harrisburg, Pa.*

If we assume the three sides of a triangle 3, 4, 5, these sides will form a right-angled triangle. If we multiply these numbers by the square number 9, the sides will be 27, 36, 45, respectively, of which the digits will in each case be 9, a square. By multiplying the sides again by 9, they will become 243, 324, 405, whose digits will in each case be $= 9 = \square$, &c. &c. And if we multiply the sides 27, 36, 45, by 4, we get 108, 144, 180, whose digits are in each case also equal to $9 = \square$. &c. &c.

QUESTION VI.—*By Mr. John Rochford, New-York.*

Find rational right-angled triangles of equal perimeters, so that the area of each will be m times a square number, and exhibit the sides of three of them in integers, such that the sum of their squares will be equal to a square.

☞ See the remark of the Editor on this question in the acknowledgments.

QUESTION VII.—*By Mr. Charles Potts, Philadelphia.*

The sides of a triangle are in arithmetical progression of which the common difference is given $= a$, and the sum of the cubes of the sides is given $= b$; to determine the sides of the triangle.

SOLUTION.—*By William F. Kells, Bergen.*

Let $x =$ the middle term, then will $x-a$ and $x+a$ be the first and last terms, and the sum of their cubes is by the question equal to b, therefore $(x-a)^3+x^3+(x+a)^3=b$
which by reduction becomes $3x^3+6a^2x=b$,
or by division $x^2+2a^2x=\frac{b}{3}$, which being resolved by the rule of Cardan for cubics, gives the value of x, and hence the extremes are easily obtained.

QUESTION VIII.—*By John Capp, Esq. Harrisburg, Pa.*

A manufacturer at the Salt-works in the state of New-York, during the late war, had on hand a thousand barrels of salt, valued at two dollars per barrel at the works, engaged a carrier who agreed to carry salt at the rate of five dollars per barrel to Pittsburg, and to take one moiety of his wages in salt at the works, and the other moiety at Pittsburg on the delivery of the residue: how much salt did the carrier receive at each place? and what was the gain or loss of the manufacturer in the transaction, supposing salt to have been rated at ten dollars per barrel at Pittsburg?

FIRST SOLUTION.—*By James Phillips, Harlem.*

Let $x =$ the number of barrels the carrier received at the salt works, then $2x$ will be their value, and this by the question is equal to half the charge on those taken to Pittsburg; therefore $2x=\frac{5000-5x}{2}$, whence $x=555\frac{5}{9}$ and hence $111\frac{1}{9}$ the number received at Pittsburg, and $\$1333\frac{1}{3}$ the gain of the manufacturer.

SECOND SOLUTION.—*By Charles Farquhar, Alexandria, Md.*

Let $x =$ the number of barrels the carrier received at the salt works, then $100-x =$ what he carried to Pittsburg, and $5(1000-x) =$ his wages in all, therefore

$$\frac{5(1000-x)}{2}\div 2=x, \text{ or } x=\frac{5000}{9};$$

therefore dividing half his wages $(=\frac{10000}{9})$ by $\$10$ we get the

salt he received at Pittsburg $=\frac{1000}{9}$; and the manufacturer's gain is now easily found $=\frac{4000}{3}=1333\frac{1}{3}$.

THIRD SOLUTION.—*By the Rev. T. Clowes, Chestertown, Md.*

Let $x=$ the number of barrels the carrier received at Salina, then $1000-x=$ the number he carried to Pittsburg, for which he is to receive $5000-5x$ dollars. Half of this sum $=2500-2\frac{1}{2}x$ dollars, or its value $1250-1\frac{1}{4}x$ barrels to be received at Salina, Hence per question $1250-1\frac{1}{4}x=x$: this reduced we have $x=555\frac{5}{9}$ barrels of salt received at Salina, worth $1111\frac{1}{9}$ $. He must therefore carry $444\frac{4}{9}$ barrels to Pittsburg, for which he must receive salt for an equal sum as at Salina $\frac{1111\frac{1}{9}}{10}=111\frac{1}{9}$ barrels received at Pittsburg. Then $555\frac{5}{9}+111\frac{1}{9}=666\frac{2}{3}$ barrels paid in all, and $1000-666\frac{2}{3}=333\frac{1}{3}$ barrels left for the manufacturer at Pittsburg, the value of which $10 is $3333\frac{1}{3}$.

If the manufacturer had engaged to pay $5, or the half of every barrel delivered at Pittsburg, he would have sold the remaining half for $5000, and thus have gained $1666\frac{2}{3}$, which by his present bargain he loses.

FOURTH SOLUTION.—*By Tyro, Lexington, Kentucky.*

Let $x=$ the quantity of salt given at the works as part payment of the carriage, we get this equation $4x=5000-5x$, whence $x=555\frac{5}{9}$ and $\frac{1}{5}$th of this received at Pittsburg $=111\frac{1}{9}$; there remains to the owner $333\frac{1}{3}$ barrels worth $3333\frac{1}{3}$. Now if he had sold the 1000 barrels at Pittsburg and paid the carriage in money, he would have received $5000, his loss is therefore $1666\frac{2}{3}$.

FIFTH SOLUTION.—*By James M'Ginnis, Harrisburg, Pa.*

If 1 represent the whole quantity retained at the works for carriage, then per question $\frac{1}{5}$ will represent the quantity taken at Pittsburg for do. and $1-\frac{1}{5}=\frac{4}{5}=$ the part carried: then $1+\frac{1}{5}:1::1000:555\frac{5}{9}$ barrels the part retained at the works, which at 2 per barrel amounted to $111\frac{1}{9}$ and $\frac{555\frac{1}{9}}{5}=111\frac{1}{9}$ the part taken at Pittsburg, for $111\frac{1}{9}\times 10=1111\frac{1}{9}$: and $1000-\overline{555\frac{5}{9}+111\frac{1}{9}}=333\frac{3}{9}$ the quantity which the merchant has clear, which at 10 dollars amounted to $3333\frac{1}{3}$ dollars which is $1333\frac{1}{3}$ gain more to the merchant than if he had taken 2 dollars per barrel at the works.

QUESTION IX.—*By Mr. William Kells, Bergen, N. J.*

It is required to find the diameter of a vessel in the form of a hemispherical surface, which would cost as much to buy it full of

wine worth a dollar per gallon as would buy a lid for the same worth six dollars per square inch.

FIRST SOLUTION.—*By Solomon Wright, Lumberville, Buck's Co. Pennsylvania.*

Let $x=$ the diameter, $p=$ the circumference of a circle whose diameter is unity, and $a=$ the number of solid inches in a wine gallon; then $px^2=$ then $px^2=$ the superficies of the sphere, and $\frac{px^3}{6}=$ the solidity of the sphere, also $\frac{px^2}{12}=$ the hemispherical solidity, and $\frac{px^3}{12a}=$ the price of the wine. Again $\frac{px^2}{4}=$ the number of square inches in the lid which multiplied by \$6 gives $\frac{6px^2}{4}$ $=$ price of the lid, hence $\frac{px^3}{12a}=\frac{6px^2}{4}$ by the question; and by clearing and division we have $x=18a$ the diameter required.

SECOND SOLUTION.—*By Charles Potts, Philadelphia.*

Let x be the diameter in inches, and putting $a=.5236$, $b=$ $.7854$, and $c=231$, we have $\frac{ax^3}{2}=$ the solidity, $\frac{ax^3}{2c}=$ the capacity in wine gallons, and $bx^2=$ the area of the lid in square inches, wherefore by the question $6bx^2=\frac{ax^3}{2c}$, or $x=\frac{12bc}{a}=4158$ inches.

THIRD SOLUTION.—*By Robert Abbot, New-York.*

Let $x=$ the diameter in inches; then per mensuration $.2618x^2$ $=$ the content of the vessel, and $.7854x^2=$ the area of the lid: now a wine gallon being equal to 231 cubic inches, we have per question $\frac{.2618x^3}{231}\times\$1=.7854x^2\times\$6$; wherefore $x=\frac{1088.5644}{.2618}$ $=4158$ inches, the diameter required.

QUESTION X.—*By Mr. S. Wright, Bucks Co. Pa.*

Required the dimensions of a right-angled triangle, the hypothenuse of which is given, the perpendicular added to twice the base being a maximum.

FIRST SOLUTION.—*By Wm. Vogdes, Edgmont, Del. Co. Pa.*

Let $x=$ the base, $y=$ the perpendicular, and $h=$ the hypothenuse, then $x^2+y^2=h^2$ and $y+2x=$ a maximum.

By taking the fluxions we have $\dot{y}+2\dot{x}=0$, or $\dot{x}=-\frac{\dot{y}}{2}$: again

$2x\dot{x}+2y\dot{y}=0$, or $\dot{x}=-\frac{y\dot{y}}{x}$, whence $-\frac{\dot{y}}{2}=-\frac{y\dot{y}}{x}$ or $x\dot{y}=2y\dot{y}$, and $x=2y$. Now substitute this value of x in the first equation, and it becomes $4y^2+y^2=h^2$, or $5y^2=h^2$, hence $y=h\sqrt{\frac{1}{5}}$, and $x=2h\sqrt{\frac{1}{5}}$, which are the perpendicular and base required.

SECOND SOLUTION.—*Mr. —— Hammond, New-York.*

Let $x=$ the perpendicular and $y=$ the base, and to avoid fractional results let the given hypothenuse be $2.236+=\sqrt{5}$. Then $x+2x=$ maximum, and by the properties of the triangle $x^2+y^2=5$. Therefore $\dot{x}+2\dot{y}=0$, because it is a maximum, and $2x\dot{x}+2y\dot{y}=0$, because it is a constant quantity, therefore $\dot{x}=-2\dot{y}$, which value, if $\dot{x}$ being substituted in the second fluxional equation gives $2y\dot{y}=4x\dot{y}$, or $y=2x$, hence $4x^2+x^2=5x^2=5$ and $x=1$ consequently $y=2$.

THIRD SOLUTION.—*By J. Ingersoll Bowditch, Boston.*

Let $a=$ the hypothenuse which is given, and $x=$ the perpendicular, then $\sqrt{a^2-x^2}$ will represent the base. The question gives $x+2\sqrt{a^2-x^2}=$ max. hence $dx-\frac{2xdx}{\sqrt{a^2-x^2}}=0$, $\therefore 2x=\sqrt{a^2-x^2}$, therefore $a^2-x^2=4x^2$, $\therefore a^2=5x^2$ $\therefore x=\frac{a}{\sqrt{5}}$ then $\sqrt{a^2-x^2}=\frac{2a}{\sqrt{5}}$ the base of the triangle.

FOURTH SOLUTION.—*By Charles Farquhar, Alex. D. C.*

Let $x=$ the base, and $h=$ the hypothenuse; then $\sqrt{h^2-x^2}+2x=$ max. or by taking the fluxion and reducing $5x^2=4h^2$, or $x=\frac{2h}{\sqrt{5}}$ and the perp. $=\frac{h}{\sqrt{5}}$, that is, the base = twice the perpendicular.

QUESTION XI. (43.)—*By Mr. Henry Darnell, Philadelphia.*

A ship came to anchor, and after paying out 40 fathoms of cable she was fifty fathoms from her buoy, which floated perpendicularly over her anchor: the depth of the water is required.

FIRST SOLUTION.—*By Robert Abbot, New-York.*

Let $x=$ the depth of water required; then, per question, and 47 Eu. I. $(x+40)^2=x^2+50^2$, hence by reduction $x=11\frac{1}{4}$ fathoms.

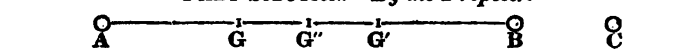
SECOND SOLUTION.—*By Mr. Hammond, New-York.*

It is evident that the ship, the anchor, and the buoy are the three angular points of a right angled triangle, whose hypothenuse exceeds the perpendicular by 40, and whose base is 50; therefore, let $x=$ the perpendicular, or depth required, then $x+40=$ the hypothenuse, and by Euc. 47, I. $x^2+2500=x^2+80x+1600$, and $80x=900$, or $x=11\frac{1}{4}$ fathoms, the depth required.

QUESTION XII. (44.)—*By Mr. James Phillips, Harlem.*

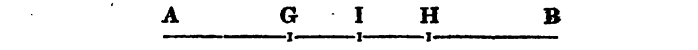
Three masses are placed by pairs at the extremities of an uniform rigid bar void of gravity, whose length is 2040 inches, in such a manner that if G be the common centre of gravity of A and B, and B be removed and C put in its place, the distance of the common centre of gravity of A and C from G will be 675 inches nearer the extremity B; but if B be now placed with A, then the distance of the common centre of gravity of A+B and C from that of A and C is 216 inches nearer the extremity A. Now if the sum of these be 20 lbs. what is the weight of each and the distance AG?

FIRST SOLUTION.—*By the Proposer.*



Let $AG=x$, $AB=a=2040$, $b=675=GG'$, and $c=675-216=459=GG''$, then $A:B::BG:AG$ $\therefore$ $x:a-x::B:A$; hence $A=\frac{a-x}{x}B$; also, as $x+b:a-x-b::C:A=\frac{a-x}{x}B$, whence $C=\frac{(a-x).(b+x)B}{x(a-x-b)}$; and finally, $A+B=\frac{a}{x}.B:C=\frac{(a-x).(b+x)B}{x(a-x-b)}::a-x-c:x+c$, from which we obtain the equation $x^3+(c+b-a)x^2-(ab-bc)x=a^2c-a^2b$, and consequently, by the known methods of resolution, $x=765$. Again, $A+B:C::BG'':AG$; or, $A+B+C\,(20):C::AB\,(2040):AG''\,(765+459)$, hence, $C=12$ and $A+B=8$; but $A+B:B::AB:AG$, whence $B=3$ and $A=5$.

SECOND SOLUTION.—*By Charles Potts, Philadelphia.*



Let the line AB represent the bar, G the centre of gravity of the masses A and B, H the centre of gravity of A and C, and I the centre of gravity of A+B and C. By the question, $AB=2040=a$, $GH=675=b$, $GI=459=c$, and $A+B+C=20=d$. As-

sume $A=x$, $C=y$, and $AG=z$. Then when A and B are applied to the bar, $x : d-x-y :: a-z : z$, from which analogy $x=\frac{(a-z)\,.\,d-(a+z)\,.\,y}{a}$. When A and C are applied, we have $x : y :: a-z-b : z+b$, or $x=\frac{(a-z-b).y}{z+b}$. And when A+B and C are applied, $d : y :: a : z+c$, or $y=\frac{(z+c)\,.\,d}{a}$. Now, substitute the value of y in each of the values of x, and equate the results, by which we obtain the equation $z^3+(c-a+b)z^2+(bc-ab)\,.\,z+(b-c)a^2=0$. From this equation we find $z=765=AG$, therefore, $y=12=C$, $x=5=A$, and $B=3$.

QUESTION XIII. (45.)—*By Professor Strong, Hamilton College, State of N. York.*

Integrate the differential equation

$$dy = \frac{dx}{(1-x^3)^{\frac{1}{3}}}$$

in finite terms if it be possible.

FIRST SOLUTION.—*By the Proposer.*

Put $(1-x^3)^{\frac{1}{3}}=ux$, and we derive, $x=\frac{1}{(1+u^3)^{\frac{1}{3}}}$, and $dx=\frac{-u^2du}{(1+u)^{\frac{4}{3}}}$, and $\frac{dx}{(1-x^3)^{\frac{1}{3}}}=-\frac{udu}{1+u^3}=\frac{du}{3(1+u)}-\frac{(2udu-du)}{6(u^2-u+1)}+\frac{du}{2(u^2-u+1)}$, $\because \int\frac{dx}{(1-x^3)^{\frac{1}{3}}}=h.l.\left(\frac{1+u}{u^2-u+1}\right)^{\frac{1}{3}}-\frac{2}{3}(P-z)$, P being the length of the arc of a quadrant, and z being the arc of a circle to the tangent $(u-\frac{1}{2})$ and radius $\frac{\sqrt{3}}{2}$, which is also the radius of the quadrant P. And the correction has been made by supposing the integral $=0$ when $x=0$, or when $u=$ infinity. By restoring the value of $u=(1-x^3)^{\frac{1}{3}}\div x$, the integral will be exhibited in terms of x.

SECOND SOLUTION.—*By Dr. Henry J. Anderson, N. York.*

1. Put $(1-x^3)^{\frac{1}{3}}=-xy$. Then $x=\frac{1}{(1-y^3)^{\frac{1}{3}}}$, $dx=\frac{y^2dy}{(y^3-1)^{\frac{4}{3}}}$, $(1-x^3)^{\frac{1}{3}}=\frac{y}{(y^3-1)^{\frac{1}{3}}}$, and $\frac{dx}{(1-x^3)^{\frac{1}{3}}}=\frac{ydy}{y^3-1}$. Or

2. Put $(1-x^3)^{\frac{1}{3}}=\frac{x}{y}$, then we shall have $\frac{dx}{(1-x^3)^{\frac{1}{3}}}=\frac{dy}{y^3+1}$.

Either of these transformed values may be easily integrated by the common methods. Thus, taking the second, and calling its integral U, we have

$$U=\int\frac{dy}{y^3+1}=\int\frac{\frac{1}{3}(y^2-y+1)+(2-y)(y+1)\}dy}{3(y^3+1)}=\frac{1}{3}\int\frac{dy}{y+1}$$
$$+\frac{1}{3}\int\frac{(2-y)dy}{y^2-y+1}=\frac{1}{3}l(y+1)-\frac{1}{3}l\sqrt{y^2-y+1}+\frac{1}{3}\sqrt{3}\left(\text{arc. tan.}\right.$$
$$\left.\frac{2y-1}{\sqrt{3}}\right)+C=\frac{1}{3}\left\{\frac{1}{2}l\frac{(y+1)^3}{y^3+1}+\sqrt{3}\,\text{arc. tan.}\,\frac{2y-1}{\sqrt{3}}\right\}+C.$$ If $U=0$, when $y=0$, or when $x=0$; in that case the corrected integral will be $U=\frac{1}{3}\left\{\frac{1}{2}l\frac{(y+1)^3}{y^3+1}+\sqrt{3}\,\text{arc. tan.}\,\frac{2y-1}{3}+\sqrt{3}\right.$ $\left.\text{arc. tan.}\,\frac{1}{\sqrt{3}}\right\}$. But arc. tan. $\frac{2y-1}{\sqrt{3}}+$ arc. tan. $\frac{1}{\sqrt{3}}=$ arc. tan. $\frac{y\sqrt{3}}{2-y}$, therefore

$$U=\int\frac{dy}{y^3+1}=\frac{1}{3}\left\{\frac{1}{2}l\frac{(y+1)^3}{y^3+1}+\sqrt{3}\,\text{arc. tan.}\,\frac{y\sqrt{3}}{2-y}\right\}$$

It may be remarked that this differential comes under the form $x^{m-1}dx(a+bx^n)^{\frac{p}{q}}$, which, as is very well known, is always integrable when $\frac{m}{n}$ is a whole number. For if we put $a+bx^n=y^q$, the differential is transformed to (3) $\frac{q}{nb}y^{p+q-1}dy\left(\frac{y^q-a}{b}\right)^{\frac{m}{n}-1}$, a rational differential when $\frac{m}{n}$ is an integer, and fractional and rational when $m=0$. Now $\frac{dx}{(1-x^3)^{\frac{1}{3}}}$ may be thus expressed, x^{-1} $dx\,(x^{-3}-1)^{-\frac{1}{3}}$ where $m=0$, $\frac{p}{q}=-\frac{1}{3}$, $a=-1$, $b=1$, $n=-3$. Here, if we take $p=1$, the expression (3) becomes $\frac{dy}{y^3+1}$, if $p=-1$, it becomes $\frac{-ydy}{y^3+1}$, or if at the same time we put $a+bx^n=-y^q$, it becomes $\frac{ydy}{y^3-1}$, the same values we had already obtained.

THIRD SOLUTION.—*By Dr. Bowditch, Boston.*

Put $x^3=\frac{z^3}{1+z^3}$, and the proposed integer becomes $\int\frac{dz}{1+z^3}$, which being *rational*, is easily solved by the rules deduced from Cotes' Theorem, and we get generally

$$\int\frac{dx}{\sqrt[3]{1-x^3}}=\int\frac{dz}{1+z^3}=\frac{1}{\sqrt{3}}\text{ arc.}\left(\tan.\frac{z\sqrt{3}}{2-z}\right)+\frac{1}{6}\,l\,\frac{1+2z+zz}{1-z\sqrt{3}+zz}$$

This log. vanishes when $z=0$ and $z=\infty$, corresponding to $x=0$ and $x=1$. The arc vanishes when $z=0$ and when $z=\infty$, $\frac{z\sqrt{3}}{2-z}=-\sqrt{3}=\tan.\,120°=\tan.\frac{2\pi}{3}$, and in this case the integral becomes $\frac{2\pi}{3\sqrt{3}}$ as given by Euler, who uses generally the transformation $x^n=\frac{z^n}{1+z^n}$ to render rational (and of course integrable) an equation of the form $\int\frac{x^{n-m-1}dx}{(1-x^n)^{\frac{n-m}{n}}}$ treated of so much by him and by Le Gendre.

QUESTION XIV. (46.)—*By Eboracensis.*

To inscribe the greatest rectangle in one of the ovals of a lemniscate, of which the equation is $a^6y^2=a^6x^2-x^8$.

FIRST SOLUTION.—*By John Rochford, N. Y.*

Put $x=az^{\frac{1}{2}}$, then $y^2=a^2(z-z^4)=a^2(z-z'^4)$ for the other abscissa; then by subtraction and dividing by a^2, $(z'-z)-(z'^4-z^4)=0$. And dividing by $z'-z$ there results $z'^3+z'z(z'+z)+z^3=1$. Now put $z'+z=s$ and $z'z=p$ and we shall have $s^3-3sp+sp=s^3-2sp=1$ and $p=\frac{s^3-1}{2s}$. Hence $z'-z=\left(s^2-\frac{4s^3-4}{2s}\right)^{\frac{1}{2}}=\left(\frac{2-s^3}{s}\right)^{\frac{1}{2}}$ and then $z'=\frac{s}{2}+\left(\frac{2-s^3}{4s}\right)^{\frac{1}{2}}$ and $z=\frac{s}{2}-\left(\frac{2-s^3}{4s}\right)^{\frac{1}{2}}$. Again,

$$x'-x=a\left(z'^{\frac{1}{2}}-z^{\frac{1}{2}}\right)=a\left(\left(\frac{s}{2}+\left(\frac{2-s^3}{4s}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}-\left(\frac{s}{2}-\left(\frac{2-s^3}{4s}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right);$$

and the square of this multiplied by $y^2=$ a max. or

$$\left(s-\left(\frac{s^3-1}{2s}\right)^{\frac{1}{2}}\right)\times\left(\frac{s}{2}-\left(\frac{2-s^3}{4s}\right)^{\frac{1}{2}}-\left(\frac{s}{2}-\left(\frac{2-s^3}{4s}\right)^{\frac{1}{2}}\right)^4\right)=\frac{4s^2-s^5}{4}-\frac{1}{4s}-\frac{4s^4-s^6-1}{4s^2}\left(\frac{s^3-1}{2s}\right)^{\frac{1}{2}}=\text{a max.}$$

which differentiated, made

$=0$, and divided then by ds, will produce the expression $32s^6-20s^8+4s^2-(-16s^7-16s^4+8s)\left(\frac{s^3-1}{2s}\right)^{\frac{1}{2}}+(2s^3+1)(4s^3-s^6-1)\times\left(\frac{2s}{s^3-1}\right)^{\frac{1}{2}}=0$. This equation solved as it is now, or freed first from surds and then solved, will give the value of s, whence all other requisites easily flow.

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SECOND SOLUTION.—*By Dr. Bowditch, Boston.*

Let x, y be two values of x, y, satisfying the equation $y^2=x^2-x^8$, a being the unit of measure, and x', y' two other values satisfying the same equation $y'^2=x'^2-x'^8$. Now the differential of the proposed equation gives $dx=\frac{ydy}{x-4x^7}$, $ydx=\frac{y^2dy}{x-4x^7}=\frac{x^2-x^8}{x-4x^7}\times dy$ which added to xdy gives $d.(xy)=\frac{2x-5x^7}{1-4x^6}.dy$, and in like manner $d(x'y')=\frac{2x'-5x'^7}{1-4x'^6}.dy'$, whose difference is $d(x'y'-xy)=\frac{2x'-5x'^7}{1-4x'^6}dy'-\frac{2x-5x^7}{1-4x^6}dy$. If we now put $y'=y$ and $dy'=dy$, the quantity $d(x'y'-xy)$ will become $d.(y.\overline{x'-x})$, which is proportional to the differential of the area which is to be a *maximum*, and is therefore 0, and dy' being $=dy$, the expression divided by it gives to find x and x' the following equations:

(1.) $$\frac{2x'-5x'^7}{1-4x'^6}=\frac{2x-5x^7}{1-4x^6}$$

(2.) and $y^2=y'^2$ gives $x'^2-x'^8=x^2-x^8$

These equations may be solved approximately in their present form, or may be reduced in various ways; for example, by putting $x'=rx$. For the equation (2.) $x'^2-x^2=x'^8-x^8$ is divisible by x'^2-x^2 and gives $1=(x'^2+x^2)(x'^4+x^4)=x^6(1+r^2)(1+r^4)$, and the equation (1.) substituting $x'=rx$ and dividing by x becomes by reduction of the form $A+Bx^6+6x^{12}=0$, A, B, C being functions of r, and substituting $x^6=\frac{1}{(1+rr)(1+r^4)}$ we obtain an equation of the 12th degree in r. I did not continue this calculation as there was no other difficulty except its length. It may be observed that the method may be used for any other equations of a similar kind, such as were solved in the former numbers as $y^2=x^2-x^4$, $y^2=x^2-x^6$, &c.

We might also use another method, putting in the proposed equation $y^2=x^2-x^8$, $y=x\cos.\phi$, which gives $x^2\cos.\phi^2=x^2-x^8$ $\therefore x^6=1-\cos.\phi^2=x\sin.\phi^2$ and $x=\sin.\phi$, their product $xy=\cos.\phi.\sin.\phi$, and then in like manner $x'y'=\cos.\phi'.\sin.\phi'$; but I did not follow this method; it is hardly so simple as the other.

THIRD SOLUTION.—*By Professor Strong, Hamilton College, State of N. York.*

Let $x \ldots x'$ denote the two abscisses corresponding to the equal values of y, (and for convenience suppose $a=1$,) then we shall have $x'^2-x'^8=x^2-x^8 \therefore x'^2-x^2=x'^8-x^8$; divide by x'^2-x^2 and we have $x'^6+x'^4x^2+x'^2x^4+x^6=1$; suppose $x'>x$ and put $x'=xz$, then by substitution we have $x^6(z^6+z^4+z^2+1)=1 \therefore x=\frac{1}{(z^6+z^4+z^2+1)^{\frac{1}{6}}} \therefore x'=zx=\frac{z}{(z^6+z^4+z^2+1)^{\frac{1}{6}}}$ and $x'-x=\frac{z-1}{(z^6+z^4+z^2+1)^{\frac{1}{6}}}$ also $y^2=x^2-x^8=\frac{1}{(z^6+z^4+z^2+1)^{\frac{1}{3}}}-\frac{1}{(z^6+z^4+z^2+1)^{\frac{4}{3}}}=\frac{z^6+z^4+z^2}{(z^6+z^4+z^2+1)^{\frac{4}{3}}} \therefore y=\frac{z(z^4+z^2+1)^{\frac{1}{2}}}{(z^6+z^4+z^2+1)^{\frac{4}{3}}}$ hence we have $y(x'-x)=\frac{(z^2-z)\times(z^4+z^2+1)^{\frac{1}{2}}}{(z^6+z^4+z^2+1)^{\frac{4}{3}}}=$ the rectangle $=$ max. hence $\frac{(z^2-z)^6\times(z^4+z^2+1)^3}{(z^6+z^4+z^2+1)^5}=$ max. Put the differential of this expression $=$ to 0, and reduce and order the terms, and then results the equation $3z^{11}-6z^{10}+4z^9-10z^8+3z^7-12z^6-12z^5+3z^4-10z^3+4z^2-6z+3=0$; divide by $z+1$ and it reduces to $3z^{10}-9z^9+13z^8-23z^7+26z^6-38z^5=26z^3-23z^3+13z^2-9z+3=0$; divide by z^5 and collect terms which have the same coefficients, and there results $3(z^5+\frac{1}{z^5})-9(z^4+\frac{1}{z^4})+13(z^3+\frac{1}{z^3})-23(z^2+\frac{1}{z^2})+26(z+\frac{1}{z})-38=0$ put $z+\frac{1}{z}=r$ then $z^2+\frac{1}{z^2}=r^2-2$ $\therefore z^3+\frac{1}{z^3}=r^3-3r \ldots z^4+\frac{1}{z^4}=r^4-4r^2+z \ldots z^5+\frac{1}{z^5}=r^5-5r^3+5r$, hence by substituting for $z+\frac{1}{z}$ $z^2+\frac{1}{z^2} \ldots z^5+\frac{1}{z^5}$, &c. their values in terms of r, in the equation above given and reducing there results the equation $3r^5-9r^4-2r^3+13r^2+2r-10=0$. This equation solved by the usual rules of approximation gives $r=2.673437$ nearly $\therefore z+\frac{1}{z}=2.673437 \therefore z^2-2.673437\times z=-1$. This quadratic solved gives $z=2.2237765$ nearly, substitute this value of z in the expressions for $x \ldots x' \ldots y$ in terms of z and we have $x=0.433189 \ldots x'=0.963316 \ldots y=0.431755 \therefore x'-x=0.530127$, and hence $(x'-x)y=0.228885$ for the value of the rectangle required. If we multiply these values of x, x', y, by a, and the value of the rectangle by a^2, we shall have the values of x, x', y, and the rectangle when a designates any number whatever.

Remark. The method of investigation which I have here used will enable us to inscribe the maximum rectangle in the general lemniscate, whose equation is $a^{2n}y^2=a^{2n}x^2-x^{2n+2}$, and the method of reasoning is the same as above, viz. by putting the two equal values of y^2 in terms of $x \ldots x'$ equal to each other, then transposing so as to have x'^2-x^2 on one side of the equation, and dividing by x'^2-x^2, and supposing that $a=1$ as above, and we shall have $x'^{2n}=x'^{2n-2}x^2+x^{2n-4}x^4+\&c. \ldots x^{2n}=1$, put then $x'=zx$, $x'<x$ and we shall have $x=\frac{1}{(z^{2n}+z^{2n-2}+z^{2n-4}+\&c \ldots +1)^{\frac{1}{2n}}}$ and $x'=\frac{z}{(z^{2n}+z^{2n-2}+z^{2n-4}+ \ldots +1)^{\frac{1}{2n}}}$ $\therefore$ we have $x'-x=\frac{z-1}{(z^{2n}+z^{2n-2}+z^{2n-4}+ \ldots +1)^{\frac{1}{2n}}}$ also $y^2=x^2-x^{2n+2}=\frac{x^{2n}+z^{2n-2}+\&c. \ldots +z^2}{(z^{2n}+z^{2n-2}+ \ldots +1)^{\frac{1}{n}}}$ $\therefore y=\frac{z(z^{2n-2}+z^{2n-4}+ \ldots +1)^{\frac{1}{2}}}{(z^{2n}+z^{2n-2}+ \ldots +1)^{\frac{1+n}{2n}}}$ hence the rectangle $=\frac{(z^2-z)\times(z^{2n-2}+z^{2n-4}+ \ldots +1)^{\frac{1}{2}}}{(z^{2n}+z^{2n-2}+ \ldots +1)^{\frac{2+n}{2n}}}=$ max. or $\frac{(z^2-z)^{2n}\times(z^{2n-2}+z^{2n-4}+ \ldots +1)^n}{(z^{2n}+z^{2n-2}+ \ldots +1)^{n+2}}=$ max. or $\frac{(z^2-z)^{2n}\times\left(\frac{z^{2n}-1}{z^2-1}\right)^n}{\left(\frac{z^{2n+2}-1}{z^2-1}\right)^{(n+2)}}=$ max. with which we are to proceed just as above. By applying this process to the lemniscate given in the first No. of the Diary, and by using the same notation as above, there results the equation $z^2-3z+1=0$, from which the same conclusions follow, as formerly deduced by very different processes.

Again, the second question, when treated in this manner, gives the equation in terms of z, as follows: $z^6-3z^5+4z^4-7z^4+4z^2-3z+1=0$; divide by z^3 and put $z+\frac{1}{z}=r$, and there results the equation $r^3-3r^2+r-1=0$; this equation solved gives r, and thence z is had from the quadratic $z^2-rz=-1$, whence x, x', y, are found, and I find by calculation that the results are the same as given in the last No. of the Diary. From what is done above, it appears that the equation in z has its terms at equal distances from the extremes, such that they have the same signs and the same co-efficients, whence the equation in z can always be reduced to the equation in r, of half the dimensions which solved gives

r, and thence x is had from the solution of the quadratic $x^2 - rz = -1$, and thence every thing else becomes known.

FOURTH SOLUTION.—*By Dr. Henry J. Anderson, New-York.*

In the solution of this question, I shall assume the required rectangle, as having its sides parallel respectively with the axes of the lemniscate, leaving it to the ingenuity of the contributors to this journal to determine whether it be not possible to inscribe rectangles obliquely in the oval of the lemniscate, and whether, (if it be possible,) the absolute maximum may not be found among these.

A slight examination of the equation $a^{2n-2}y^2 = a^{2n-2}x^2 - x^{2n}$ is sufficient to show that $-a$ and $+a$ are the limits of the values of x, and $-aFn$ and $+aFn$, where $Fn = \sqrt{n^{\frac{1}{1-n}} - n^{\frac{n}{1-n}}}$, the limits of the values of y; that for every value of x between its limits, there are two corresponding values of y, one positive and one negative equal to it, and four for every value of y between its limits, two positive and two negative equal to them, and no others. Now let y be any ordinate, and x and x_i be the two positive or two negative corresponding abscissæ, then we have by the equation of the curve

$$a^{2n-2}y^2 = a^{2n-2}x^2 - x^{2n}$$
$$a^{2n-2}y^2 = a^{2n-2}x_i^2 - x_i^{2n}$$

Subtracting the second of these equations from the first, and dividing by $x_i^2 - x^2$ we obtain

[1] $x_i^{2n-2} + x_i^{2n-4}x^2 + x_i^{2n-6}x^4 + x_i^{2n-8}x^6 + \&c. = a^{2n-2}$

Transposing x_i^{2n-2}, and multiplying by x_i^2 we have

[2] $x_i^2(x_i^{2n-4} + x_i^{2n-6}x^2 + x_i^{2n-8}x^4 + \&c.) = a^{2n-2}y^2$

Now put $x_i x = p$, and $x^2 + x^2 = xp$. Then by known formulæ, we have

[3] $p^{n-1}(x^{n-1} - A_1x^{n-3} + A_2x^{n-5} - A_3x^{n-7} + \&c.) = a^{2n-2}$

[4] $p^n(x^{n-2} - B_1x^{n-4} + B_2x^{n-6}y - B_3x^{n-8} + \&c.) = a^{2n-2}y^2$

where $A_1 = \frac{n-2}{1}$, $A_2 = \frac{n-3}{1}\frac{n-4}{2}$, $A_3 = \frac{n-4}{1}\frac{n-5}{2}\frac{n-6}{3}$, &c.

$B_1 = \frac{n-3}{1}$, $B_2 = \frac{n-4}{1}\frac{n-5}{2}$, $B_3 = \frac{n-5}{1}\frac{n-6}{2}\frac{n-7}{3}$, &c.

Hence $d[3] = 0$, and by the question $(x_i - x)^2 \times [4] = \text{max.}$ or

[5] $p^{n+1}(z-2)(x^{n-2} - B_1x^{n-4} + B_2x^{n-6} - \&c.) = \text{max.}$

Differentiating the equations [3] and (5), equating the values of $\frac{dp}{dz}$ and reducing, we obtain the final equation,

[6] $C_1x^{2n-8} - 2C_2x^{2n-4} - C_3x^{2n-5} + 2C_4x^{2n-6} + C_5x^{2n-7}$
$- 2C_6x^{2n-8} - C_7x^{2n-9} + 2C_8x^{2n-10} + \&c. = 0$

where C_1, C_2, &c. may be easily calculated thus: let m be any positive integer, then

$$C_{2m-1}=MB_{m-1}+M'A_1B_{m-2}+M''A_2B_{m-3}\ldots\ldots M^{(m-1)}A_{m-1}$$

where $A_0=1$, $B_0=1$, $M=2m(n-1)$, and M, M', M'' &c. terms of a decreasing arithmetical progression whose difference is $4n$; and

$$C_{2m}=NB_{m-1}+N'A_1B_{m-2}+N''A_2B_{m-3}\ldots\ldots N^{(m-1)}A_{m-1}$$

where $N=(2m+1)(n-1)$, and N, N', N'', &c. are terms of a similar progression. Having obtained the value of x from the equation (6), the equation (3) gives the value of p, and equation (4) the value of y; and x, and x are given by the equations

$$(x,+x)^2=(x+2)p$$
$$(x,-x)^2=(x-2)p$$

For example:—

	Lemniscate.	*Final equation.*
(1)	$a^2y^2=a^2x^2-x^4$	$x-3=0$
(2)	$a^4y^2=a^4x^2-x^6$	$x^3-3x^2+x-1=0$
(3)	$a^6y^2=a^6x^2-x^8$	$3x^5-9x^4-2x^3+13x^2+2x-10=0$
(4)	$a^8y^2=a^8x^2-x^{10}$	$2x^7-6x^6-5x^5+20x^4+2x^3-18x^2+4x-4=0$

Then supposing $a=1$ we shall find

	x	p	x	x,	y
(1)	3.	.3333333	.3568421	.9341723	.3333333
(2)	2.7692926	.3872312	.4065902	.9523867	.4009958
(3)	2.6734380	.4173025	.4331945	.9633146	.4317607
(4)	2.6259976	.4352451	.4484893	.9704685	.4481223

QUESTION XV. OR PRIZE QUESTION.—*By Eboracensis.*

On one of the extensive plains in the state of Illinois there rises a hill of great beauty, and perfect uniformity of curvature. At four different points on the plane of its base, at different distances from it the angles of elevation of the hill were accurately determined by observation, and the mutual distances of the places of observation were ascertained by the most approved methods of admeasurement and calculation; with these data, the determination of the perpendicular altitude of the hill above the plane of its base is respectfully submitted to mathematicians for their investigation.

PRIZE SOLUTION.—*By Dr. Henry J. Anderson, N. York.*

This question is evidently the same with that which would require the radius and co-ordinates of the centre of the sphere which touches at the same time externally four given right cones, whose vertices are in the same plane, and whose axes are perpendicular to the plane.

Let $\alpha, \beta, \alpha', \beta'$, &c. be the co-ordinates of the vertices, or of the places of observation; m, m', &c. the cotangents, and n, n', &c. the cosecants of the generating angles, or what is the same, the tangents and secants of the angles of observation; put r = radius of the sphere, and x', y', z', the co-ordinates of the centre of the sphere, then will the equations of the conic surfaces be

$$z = m\sqrt{(x-\alpha)^2+(y-\beta)^2} = mu$$
$$z = m'\sqrt{(x-\alpha')^2+(y-\beta')^2} = m'u'$$
$$z = m''\sqrt{(x-\alpha'')^2+(y-\beta'')^2} = m''u''$$
$$z = m'''\sqrt{(x-\alpha''')^2+(y-\beta''')^2} = m'''u''',$$

and $(x-x')^2+(y-y')^2+(z-z')^2 = r^2$ will be the equation of the surface of the sphere. Now the loci of the centres of the spheres which touch the given conic surfaces are easily determined from the usual conditional equations of tangency, and are obviously four other conic surfaces whose equations are

$$z = mu \pm nr$$
$$z = m'u' \pm n'r$$
$$z = m''u'' \pm n''r$$
$$z = m'''u''' \pm n'''r$$

If we wish to determine the sphere which is tangent to the four given conic surfaces at once, we must consider x, y, z, r, as the same in these last equations, and the co-ordinates thus determined will be x', y', and z'. Taking, in these equations, the double sign, *minus* only, to suit the case in which the sphere is touched superiorly by the upper semi-cones, (observing, however, that if the double sign were taken *plus* only, it would give the same result,) and eliminating r and z we obtain

$$(1) \quad mu(n'-n'')+m''u''(n-n') = m'u'(n-n'')$$
$$(2) \quad mu(n'-n''')+m'''u'''(n-n') = m'u'(n-n''')$$
$$(3) \quad mu(n''-n''')+m''u''(n-n''') = m'''u'''(n-n'')$$
$$(4) \quad m'u'(n''-n''')+m''u''(n'-n''') = m'''u'''(n'-n'')$$

of which (3) (4) are contained implicitly in (1) (2). Simplifying the given co-efficients in (1) (2) we have

$$(5) \quad \mu u+\mu'u' = \mu''u''$$
$$(6) \quad \nu u+\nu'u' = \nu''u''$$

When the values of u, u', &c. are substituted in these expressions, we obtain, after reduction, two equations of the fourth order between x and y. These determine x and y, and as we have already u, u', &c. in terms of these quantities, and $r = \frac{mu-m'u'}{n-n'}$, and $z = \frac{nm'u'-n'mu}{n-n'}$, the three co-ordinates of the centre of the sphere are determined, as also the height of the hill which is $= z+r$.

Obs. 1. In order to facilitate the reduction of the two equations (5) (6), we may observe that they may be first put into these forms;

$$(\mu u)^4+(\mu' u')^4+(\mu'' u'')^4=2\{(\mu\mu' uu')^2+(\mu\mu'' uu'')^2+(\mu'\mu'' u'u'')^2\}$$
$$(\nu u)^4+(\nu' u')^4+(\nu'' u'')^4=2\{(\nu\nu' uu')^2+(\nu\nu'' uu'')^2+(\nu'\nu'' u'u'')^2\}$$

and then the values of u^2, u'^2, u^4, u'^4, &c. which are rational, may be substituted. When the operation is performed the equations will be of these forms:

$$(7)\quad A(x^2+y^2)^2+(Bx+Cy+D)(x^2+y^2)+Ex^2+Fxy+Gy^2+Hx+Iy+J=0$$

$$(8)\quad A_{,}(x^2+y^2)^2+(B_{,}x+C_{,}y+D_{,})(x^2+y^2)+E_{,}x^2+F_{,}xy+G_{,}y^2+H_{,}x+I_{,}y+J_{,}=0$$

A, B, $A_{,}$, $B_{,}$ &c. being functions of m, m', n, n', &c. and α, α', β, β', &c.

These equations may be considered as representing two lines of the fourth order, whose intersections, beside giving the projection of the centre of the sphere we have considered, furnish also the projection of the centre of the sphere which is tangent to the four cones internally, the projections of the centres of the spheres situated symmetrically with these, with respect to the semi-cones beneath the plane, being coincident with the others.

In the foregoing investigation we have supposed the centre of the co-ordinates not coincident with either of the conic vertices. It is evident, however, that if we suppose, for instance, $\alpha''=0$, $\beta''=0$, we shall somewhat simplify the co-efficients of the equations (7) (8) without diminishing their generality. It will then be found that

$$\begin{aligned}
E&= 4(\alpha\mu^2-\alpha'\mu'^2)^2\\
F&= 8(\alpha\mu^2-\alpha'\mu'^2)(\beta\mu^2-\beta'\mu'^2)\\
G&= 4(\beta\mu^2-\beta'\mu'^2)^2\\
H&=-4(\alpha\mu^2-\alpha'\mu'^2)(\gamma^2\mu^2-\gamma'^2\mu'^2)\\
I&=-4(\beta\mu^2-\beta'\mu'^2)(\gamma^2\mu^2-\gamma'^2\mu'^2)\\
J&= (\gamma^2\alpha^2-\gamma'^2\alpha'^2)
\end{aligned}$$

where $\gamma^2=\alpha^2+\beta^2$, $\gamma'^2=\alpha'^2+\beta'^2$. Here we have $F^2=4EG$, $H^2=4EJ$, $I^2=4GJ$. Therefore putting E'^2, F'^2, G'^2, in place of E, G, and J, the equation becomes

$$(9)\quad A(x^2+y^2)^2+(Bx+Cy+D)(x^2+y^2)+(E'x+F'y+G')^2=0$$

Dividing by A and putting $B=aA$, $C=bA$, &c. we have

$$(10)\quad (x^2+y^2)^2+(ax+by+c)(x^2+y^2)+(dx+ey+f)^2=0$$

This may be still further simplified, either by shifting the axis or by removing again the centre of the co-ordinates. For the

first put $x=s\cos\theta-t\sin\theta$, $y=s\sin\theta+t\cos\theta$; develope and put $d\tan\theta=e$. Then resuming the letters x and y we have

$$(11)\quad (x^2+y^2)^2+(a_{,}x+b_{,}y+c)(x^2+y^2)+(d_{,}x+f)^2=0$$

where $a_{,}=a\cos\theta+b\sin\theta$, $b_{,}=a\sin\theta-b\cos\theta$, $d_{,}=d\sec\theta$, and $\tan\theta=\frac{e}{d}=\frac{F'}{E'}=\frac{I}{H}=\frac{\beta\mu^2-\beta'\mu'^2}{\alpha\mu^2-\alpha'\mu'^2}$. Or secondly, in the equation (10) put $x=s+\alpha_{,}$, $y=t+\beta_{,}$ develope, put the co-efficient of $st=0$ and the coefficients of s^2 and t^2 equal to each other, and there results, resuming x and y,

$$(12)\quad (x^2+y^2)^2+(a_{,,}x+b_{,,}y+c_{,,})(x^2+y^2)+d_{,,}x+e_{,,}y+f_{,,}=0$$

with the equations of condition

$$(13)\quad 4(\alpha_{,}^2-\beta_{,}^2)+2(a\alpha_{,}-b\beta_{,})+d^2-e^2=0$$

$$(14)\quad 4\alpha_{,}\beta_{,}\quad +\ (a\beta_{,}+b\alpha_{,})+de=0$$

These furnish equations in $\alpha_{,}$ and $\beta_{,}$ of the fourth order. That they have each at least two real roots will easily appear, by observing that (13) is the equation of a rectangular hyperbola, having its *axes* parallel to the axes of the curve (12), and that (14) is the equation of another rectangular hyperbola, having its *asymptotes* parallel to the same. The co-ordinates of the centre of the first are $\frac{1}{4}a$, and $\frac{1}{4}b$, of that of the second, the same quantities negative. From the nature and position of these hyperbolas they must evidently have at least two points of intersection; so that it appears that the conic vertices are not the only foci of the curve, but that there are others to which the curve may be more conveniently referred.

The equation (12) may be still further simplified in form, by putting $x^2+y^2=\rho^2$, $x=\rho\cos\phi$, $y=\rho\sin\phi$; we then obtain

$$(15)\qquad \cos\phi+\rho'\sin\phi=\rho''$$

ρ' and ρ'' being given functions of ρ. This last equation, (which is perfectly general,) enables us to describe the curve (7) by points found by rule and compass. The curve (8) may be similarly constructed, and the intersections thus obtained.

Obs. 2. The values of B and C in the equation (7) when $\alpha''=0$, $\beta''=0$, are

$$B=-4\{\alpha\mu^4+\alpha'\mu'^4-(\alpha+\alpha')\mu^2\mu'^2-\alpha\mu^2\mu''^2-\alpha'\mu'^2\mu''^2\}$$

$$C=-4\{\beta\mu^4+\beta'\mu'^4-(\beta+\beta')\mu^2\mu'^2-\beta\mu^2\mu''^2-\beta'\mu'^2\mu''^2\}$$

Performing the operation indicated by $BI-CH$, we shall find, after reduction,

$$BI-CH=32G'(\alpha\beta'-\alpha'\beta)\mu^2\mu'^2\mu''^2$$

Now when the vertices $(\alpha\beta)$ $(\alpha'\beta')$ are in a straight line with that which is the origin of the co-ordinates, we have the relation $\alpha\beta'-\alpha'\beta=0$, and therefore in that case, $BI=CH$, or $BF'=CE'$, or $ae=bd$. But in the equation (11), we have $d\tan\theta=e$, therefore

$a \tan \theta = b$. Hence $a_{,} = a \sec \theta$, and $b_{,} = 0$, and the equation (11) becomes

$$(16)\quad (x^2+y^2)^{\cdot}+(a_{,}x+c)\,(x^2+y^2)+(d_{,}x+f)^2=0$$

Moreover, $\tan\theta = \dfrac{\beta\mu^2-\beta'\mu'^2}{\alpha\mu^2-\alpha'\mu'^2}$, which, when $\alpha\beta'-\alpha'\beta=0$, becomes $= \dfrac{\beta}{\alpha}$ or $\dfrac{\beta'}{\alpha'}$. Hence it appears that the new axis is the straight line in which the three vertices are situated. We should have immediately obtained the same result, if in equation (9) we had put $\beta=0$, $\beta'=0$, for then $C=0$, $F'=0$, and

$$(17)\quad (x^2+y^2)^2+(ax+c)\,(x^2+y^2)+(dx+f)^2=0$$

which is the same that (16) becomes when α and α' are referred to the new axes. If the vertex $(\alpha'''\beta''')$ be also in the same straight line, then equation (8) becomes

$$(18)\quad (x^2+y^2)^2+(a'x+c')\,(x^2+y^2)+(d'x+f')^2=0$$

where x and y are referred to the same axes as in (17). Here the elimination is effected with the greatest ease. Subtracting (18) from (17), and substituting the obtained value of x^2+y^2 in either equation, there results, after putting

$$\frac{2(d'f'-df)}{d^2-d'^2}=p,\ \frac{f'^2-f^2}{d^2-d'^2}=q,\ \frac{a'-a}{d^2-d'^2}=p',\ \frac{c'-c}{d^2-d'^2}=q'$$

$$(19)\quad (x^2+px+q)^2+(ax+c)\,(p'x+q')\,(x^2+px+q) +(dx+f)^2(p'x+q')^2=0$$

and $(20)\quad y^2=\dfrac{x^2+px+q}{p'x+q'}-x^2$. From which it appears (what is evident, indeed, from the symmetrical disposition of the conic surfaces,) that for each positive value of y, there is in this case an equal negative one.

Obs. 3. If the projection of the centre of the hill upon the plane of its base be in a straight line with the places of observation, then in the equation (17) we have $y=0$, and three observations are sufficient. Eq. (17) becomes a biquadratic in x, which seems strange at first, as the case seems now the common one of describing a circle tangent to three given straight lines, for which simple equations are sufficient. A little attention, however, will show that eq. (17) becomes in this case identical with

$$\mu\sqrt{[\pm(x-\alpha)]^2}+\mu'\sqrt{[\pm(x-\alpha')]^2}+\mu''\sqrt{[\pm x]^2}=0$$

which is equivalent to four simple equations, each giving a value for x. That this should be so, will appear manifest from the manner in which the eq. (7) was obtained, by the double multiplication of eq. (5), as well as from the consideration that by putting $y=0$, the equations to the three cones were changed to equations to *six* straight lines instead of three. And we may observe, that the

four values of x are not, as might at first be thought, the abscissæ to the centres of the four circles which may touch the same three straight lines, but to the centres of eight circles touched inferiorly or superiorly by the six straight lines which are the intersections of the conic surfaces with the plane of their axes. The same remarks will apply to the number and nature of the spheres in the general problem. The final equation does not contain the abscissæ or ordinates of the centres of all the spheres which touch simultaneously the conic surfaces, but only of those which touch at once either four concavities or four convexities. For the others, equations the same in form but differing in the value of the co-efficients will be necessary.

Obs. 4. When μ, μ', or $\mu''=0$, μ' for instance, then eqs. (5 & 6) become

$$(20)\quad \mu u=\mu'' u'', \qquad (21)\quad \nu u=\nu'' u''$$

which are equations to circles, except when $\mu=\mu''$ and $\nu=\nu''$, and in eq. (7) we find (in addition to the similar relations between the co-efficients arising from $\alpha''=0$, $\beta''=0$,) $B^2=4AE$, $C^2=4AG$, $D^2=4AJ$, whence eq. (7) becomes

$$\{\sqrt{A}(x^2+y^2)^2+\sqrt{E}\,x+\sqrt{G}\,y+\sqrt{J}\}^2=0$$

or, substituting the values of d, e, &c.

$$(22)\quad x^2+y^2+dx+ey+f=0$$

Now as $\mu'=m'(n-n'')$, we have the two cases of $m'=0$, $n-n''=0$. In the first case, one of the cones comes to coincide with the xy plane, and the data are three right cones and the plane of their vertices. We have also in this case $n=\pm 1$, $z=\pm r$, and equation (7) becomes the equation to a circle whose radius $=\frac{\mu\mu''}{\mu^2-\mu''^2}\gamma$, and the co-ordinates of whose centre are $\frac{\mu''}{\mu^2-\mu''^2}\alpha$ and $\frac{\mu''}{\mu^2-\mu''^2}\beta$.

The same taking place in eq. (8), it follows that x and y are determined by the intersections of two circles, and are found to be

$$x=\frac{-\delta'\pm\sqrt{\delta'^2-4\delta'\zeta''}}{2\delta'}$$

$$y=\frac{-\epsilon''\pm\sqrt{\epsilon''^2-4\delta'\zeta''}}{2\delta''}$$

where $\delta'=\delta^2+\epsilon^2$, $\epsilon'=2\epsilon\zeta-d\delta\epsilon+c\delta^2$, $\zeta'=\zeta^2-d\delta\zeta+f\delta^2$, $\delta=d-d'$ $\epsilon=e-e'$ $\zeta=f-f''$, and ϵ'' and ζ'' the same as ϵ' and ζ' changing ϵ into ζ and vice versa.

In the second case, where $n-n''=0$, we have $n=n''$, $m=m''$, and the equation (20) becomes $u=u''$, the equation to a plane perpendicular to the xy plane. And eq. (22) becomes

$$y=-\frac{\alpha}{\beta}x+\frac{\lambda^2}{\beta}$$

the equation to a vertical plane bisecting at right angles the straight line which joins the two vertices (o o) and $(\alpha\beta)$.

This is the case when two of the angles of observation are equal. If only two are equal, x and y are determined by the intersections of a straight line with the curve (7) or (8). If the four have two equalities, then x and y are determined by the intersection of two straight lines, or two vertical planes, the equation to one of which is given above. This is moreover manifest from geometrical considerations.

Obs. 5. If $m=\infty$, that is, if one of the observed angles be $=90°$, then one of the cones becomes a vertical straight line, and eq. (7 and 8) become equations to circles.

Obs. 6. If the hill be given a hemisphere, then three observations are sufficient, and eqs. (7 and 8) again become equations to circles.

Obs. 7. There are a great variety of other cases which it would be interesting to examine, for instance, where r is given, or z, or $z+r$, with three cones; in the latter case, to determine r, z, u, &c.; or the places of observation might be given by three co-ordinates instead of two, in which case, two equations in x and y of the eighth degree would be obtained.

Obs. 8. It would take much time to give even a general idea of the characters of the curves embraced in the equations (7) and (8), as they vary exceedingly with the variations of the data of the question.

ACKNOWLEDGMENTS, &c.

The following ingenious gentlemen favoured the editor with solutions to the questions in Article VII. No. 2. The figures annexed to the names refer to the questions answered by each as numbered in that article.

It is necessary to observe that the VIth question being unintelligible is excluded from the list of acknowledgments; so that the omission of the figure 6, in the numbers annexed to the names, is not to be considered as implying any deficiency or failure in the number of solutions furnished by Correspondents.

Dr. Bowditch, of Boston, Professor Strong, of Hamilton College, Dr. Henry J. Anderson, N. York, John Rochford, Amelia Co. Virginia, and S. Y. of Albany, state of N. York, each most ingeniously answered all the questions.

Charles Farquhar, Alexandria, D. C. answered all but the 13th. Benjamin Hallowell, Alexandria, D. C. answered all but 13, 15. James Phillips, all but 13, 14. Rev. Dr. Clowes, Chestertown, Maryland, J. Ingersoll Bowditch, of Boston, Farrel Ward, N. Y. James Hamilton, Trenton, William Forrest, N. Y. Martin Roche, Philadelphia, and Barclay Waterman, Philadelphia, each answered all but 13, 14, 15. John Capp, Esq. Harrisburg, all but 10,

13, 14. James McGinnis, 1, 2, 3, 4, 5, 7, 8, 9, 11, 12. David S. Bogart, New-York, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11. Cornelius Davis, Dutchess Co. N. Y. 1, 2, 3, 4, 5, 7, 9, 10, 12. Mr. —— Hammond, N. Y. 1, 2, 3, 5, 7, 8, 9, 10, 11. Charles Potts, Phila. 1, 2, 3, 4, 8, 9, 10, 11, 12. Tyro, Lexington, Ken. 1, 2, 3, 4, 7, 8, 9, 10, 11. Robert Abbat, N. York, 1, 2, 3, 4, 8, 9, 10, 11. Henry Darnall, Phila. 1, 2, 3, 4, 7, 8, 9, 10, 11. Wm. Kells, Bergen, N. J. 2, 3, 7, 8, 9, 10, 11. Mr. Kean, 2, 3, 4, 7, 8, 9, 11. Michael Floy, Jun. N. York, 1, 2, 3, 4, 7, 11. Frederick Noll, Harlem, 1, 2, 3, 7, 9, 11. Edward Giddins, Fort Niagara, state of N. York, 2, 3, 7, 8, 9, 11. Wm. Vogdes, Edgemont, Del. Co. Penn. 1, 2, 3, 4, 7, 10. John F. James, Trenton, 2, 4, 8, 9, 11. E. H. Rockwell, Fredericktown, Maryland, 2, 3, 8, 9, 11. Jesse Willets, Maiden-Creek, Berks Co. Penn. 1, 2, 3, 7, 11. Solomon Wright, Lumberville, Bucks Co. Penn. 2, 3, 9, 10. Michael Doyle, Washington College, Chestertown, Maryland, 1, 2, 3. 4. S. of Brooklyn, L. I. 1, 2, 3. Daniel Shanley, Charleston, S. C. 1, 2, 3. Patrick Byrne, Philadelphia, 1, 10. William Lenhart, 4. James Denovan, N. York, 8. James O'Farrel, N. York, 2.

The PRIZE has been awarded to Dr. Henry J. Anderson, N. Y.

The solution to the prize question by Professor Strong, was extremely ingenious and perfectly general, containing several interesting remarks on the geometry and analysis of the question.

The letters of E. Giddins, F. Ward, David S. Bogart, N. York, and S. Y. of Albany, came too late to be used in the selections of solutions.

In the list of answers published in No. III. the name of Mr. Dennis Leonard, was inadvertantly omitted. He answered the questions numbered 1, 2, 3, 6, 11, 14. Mr. Aiken's letter of solutions came to hand too late to be further noticed.

ARTICLE IX.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN Nº. V.

QUESTION I. (54)—*By Mr. Farrell Ward, N. York.*

Given $\left.\begin{array}{l} x^2+y^2 = 101.8125 \\ (x+y)x = 106.875 \end{array}\right\}$ To find the values of x and y.

QUESTION II. (55)—*By Mr. Solomon Wright, Bucks Co. Pa.*

Suppose £100 to be put to interest at 6 per cent simple interest, and at the same time £50 is placed at compound interest at the same rate: in what time will the two sums become equal?

QUESTION III. (56)—*By Tyro, Lexington, Kentucky.*

To find the values of x, y, z, from the equations
$z\times(x+y+z)=6$, $y\times(x+y+z)=12$, $x(x+y+z)=18$.

QUESTION IV. (57)—*By Mr. Michael Floy, N. York.*

There are three numbers in harmonical proportion, the sum of the first and third is 18, and the product of the three numbers is 576. Quere, the numbers.

QUESTION V. (58)—*By Tyro, Lexington, Kentucky.*

A person is entitled to receive an annuity of a certain sum forever: now the ratio of the present value of each payment to that preceding, is equal to the first payment. and it is required to determine the value of each payment when due, and the present value of the whole perpetuity, allowing discount at $\frac{6}{13}$ per cent of the present value of the perpetuity.

QUESTION VI. (59)—*By John Capp, Esq. Harrisburg, Penn.*

A and B purchased a valuable farm, containing 900 acres of land, at the rate of $2 per acre, which they paid equally between them; but on dividing the same, A got that part of the farm which contained the best of the improvements, and agreed to pay 45 cents per acre more than B, how many acres had each, and at what price?

QUESTION VII. (60)—*By Mr. Henry Darnall, Philadelphia.*

Ye sons of science tell me when,
Between the hours of twelve and one,
My watch's hands themselves will place,
To form an angle of such space,
If its cotangent you prepare,
And of its secant get the square,
Their product shall in numbers come
To what you style a minimum?

QUESTION VIII. (61)—*By Mr. Wm. F. Kells, Bergen, N. J.*

The length of the chord of an arc of a circle being given, and the length of a perpendicular let fall on the arc from a given point in the chord, it is required to determine the length of the radius.

QUESTION IX. (62)—*By Mr. Farrell Ward, N. York.*

To find the area and length of a curve whose nature is such that the subtangent is always equal to the rectangle of the ordinate and subnormal.

QUESTION X. (63)—*By Edward Ward, New-York.*

Writers on navigation say that Middle Latitude Sailing is incorrect, and that Mercator's Sailing is strictly true: now in sailing from New-York to Gibraltar, whether is it safer to compute the daily difference of longitude by middle latitude, or Mercator's Sailing?

QUESTION XI. (64)—*By Martin Roche, Philadelphia.*

It is required to determine the place of a planet in its elliptical orbit about the sun in one of the foci, when the angular velocity of the planet about the other focus is a maximum or minimum.

QUESTION XII. (65)—*By Tyro, Lexington, Kentucky.*

In the collision of unelastic bodies the relative velocity divided in the ratio of the masses, gives the velocities lost and gained, and consequently the velocities after impact, that is, the part proportional to the greater mass is the velocity lost or gained by the lesser mass; and if we divide twice the relative velocity in the same manner, we obtain similar results for the collision of elastic bodies: the demonstration is required.

QUESTION XIII. (66)—*By Diarius, N. Y.*

Required the figure of a rectangular parallelopipedon, such that the sum of the edges, the sum of the faces, as well as each edge and each face, may be expressed by numbers which are rational squares.

QUESTION XIV. (67)—*By Professor Strong, Hamilton College, state of New-York.*

Integrate the equation $\frac{d\varphi}{1 \pm n \sin.^2\varphi}$.

QUESTION XV. (68)—**OR PRIZE QUESTION.**

By Farrand Benedict, Montezuma, state of New-York.

The vertical angle of a triangle is given in magnitude, and the vertex of the triangle is in the circumference of a circle given in magnitude and position; also the base of the triangle is in a straight line given in position, and one extremity of the base is given in position. It is required to determine the triangle when its base is a minimum.

THE

MATHEMATICAL DIARY,

NO. V.

BEING THE PRIZE NUMBER OF MESSRS. CHARLES FARQUHAR, ALEXANDRIA, D. COLUMBIA; J. SWALE, LIVERPOOL, ENGLAND; AND BENJAMIN HALLOWELL, ALEX. D. C.

ARTICLE X.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE IX. NO. IV.

QUESTION I.—*By Mr. Farrell Ward, New-York.*

Given $x^2 + y^2 = 101{\cdot}8125$ and $(x+y)\,x = 106{\cdot}875$ } To find the values of x and y.

FIRST SOLUTION.—*By William A. W. Stigleman, Harrisburg, Penn.*

From the last equation, we have $y = \dfrac{106{\cdot}875 - x^2}{x}$;

then, by substituting this value for y in the first equation, we have

$$x^2 + \frac{11422{\cdot}165625 - 213{\cdot}750\,x^2 + x^4}{x^2} = 101{\cdot}8125;$$

$$2x^4 - 315{\cdot}5625x^2 = -11422{\cdot}165625.$$

Hence, by quadratics, we have

$$x^2 = 56{\cdot}25, \text{ and } x = 7{\cdot}5;$$

$$\therefore y = \frac{106{\cdot}875 - x^2}{x} = 6{\cdot}75.$$

SECOND SOLUTION.—*By Mr. Daniel Shanley, Charleston, S. C.*

Let $101{\cdot}8125=a$, $106{\cdot}875=b$, and put $x=vy$; then, by substitution, we have

$$v^2y^2+y^2=a, \text{ and } v^2y^2+vy=b;$$

$$\therefore y^2=\frac{a}{v^2+1}, \text{ and } y^2=\frac{b}{v^2+v}.$$

Hence, $\frac{b}{v^2+v}=\frac{a}{v^2+1}$;

clearing of fractions,

$$bv^2+b=av^2+av;$$

by transposition,

$$(b-a)\,v^2-av=-b;$$

$$\text{or } v^2-\frac{a}{b-a}.v=-\frac{b}{b-a};$$

$\therefore$ by completing the square, extracting the root, &c.

$$v=\frac{a}{2(b-a)}\pm\frac{1}{2(b-a)}\sqrt{a^2+4ab-4b^2},$$

$$\text{or } v=\frac{a\pm1}{2(b-a)}\sqrt{(a^2+4ab-4b^2)}.$$

Hence, v being known, the values of x and y are easily found.

This question was solved in a similar manner by Messrs. Giddins, Nemo, and Sweeny.

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THIRD SOLUTION.—*By Mathetus, Bucks Co. Penn.*

By putting the given numbers equal to a and b respectively, and their difference equal to c, the difference of the equations will be expressed by $xy-y^2=c$, and from the first equation, $x=\sqrt{(a-y^2)}$. This value being substituted in the equation just obtained, we shall have $y\sqrt{(a-y^2)}-y^2=c$: which equation being reduced, y is found equal to $\sqrt{\left(\frac{a-2c}{4}\pm\frac{\sqrt{(a^2-4ac-4c^2)}}{4}\right)}=6{\cdot}75$, and x is also easily obtained$=7{\cdot}5$.

—

FOURTH SOLUTION.—*By Mr. Devoor V. Burger, N. York.*

First, let $a=$ the decimal $\cdot5625$, which is the com-

mon measure of the value of each equation, then $x^2 + y^2 = 181a$, and $(x + y)\ x = 190a$; and equalizing these equations, i. e. $\frac{x^2+y^2}{181} = \frac{(x+y)x}{190}$ reduced, &c. becomes $9x^2 - 181xy = -190y$, by comp. sq. $9x^2 - 181xy + \frac{32761y^2}{36} = \frac{25921y^2}{36}$, by extracting square root we have $3x - \frac{181y}{6} = \frac{161y}{6}$, from which $3x = \frac{181y}{6} + \frac{161y}{6}$, and $x = \frac{10y}{9}$ or $57y$. From the first eq. $x = \sqrt{181a - y^2}$, equalizing these equations, by taking the 1st or negative value of x in the preceding equation, i. e. $\left(\frac{10y}{9} = \sqrt{181a - y^2}\right)$ or $\left(\frac{100y^2}{81} = 181a - y^2,\right)$ which reduced, and cancelled, gives $y^2 = 81a$, or $y = \sqrt{81a} = 6{\cdot}75$, and $x = \frac{10y}{9} = 7{\cdot}5$.

The solutions of Messrs. Hamilton and Maccully, are nearly similar to this.

QUESTION II.—*By Mr. Solomon Wright, Bucks Co. Penn.*

Suppose £100 to be put to interest at 6 per cent. simple interest, and at the same time, £50 is placed at compound interest at the same rate: in what time will the two sums be equal?

FIRST SOLUTION.—*By Mr. James Hamilton, Trenton.*

Let $p = 50$, $r = {\cdot}06$, $R = 1{\cdot}06$, and $t =$ time;
then, $2p + 2prt = pR^t =$ the amount,
or $2 + {\cdot}12t = R^t$.
Hence, by resolving this exponential, we shall find $t = 29{\cdot}301$ years.

SECOND SOLUTION.—*By Mr. J. Ingersoll Bowditch, Boston.*

This problem may be resolved by Double Position, as follows:

Suppose 29 and 30 years for the time; then we find that in 29 years the difference between the two amounts is 3·081 too little, and that in 30 years it is 7·1745 too much. These errors being placed against their respective years, and multiplied as the rule directs, give for the sum of their products 300·4905; this divided by the sum of the errors 10.2555, gives 29·3 years for the answer.

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THIRD SOLUTION.—*By Mr. Devoor V. Burger, New-York.*

Let x=time; then, $50\times(1\cdot06)^x=100+6x$, per question.

By Logarithms, we have

$$\log.\ 50+x\times\log.\ 1\cdot06=\log.\ (100+6x).$$

By inspection, we find that the value of x is between 29 and 29½ years; ∴ taking $x=29$, log. 1·06 = ·02531, and log. 50 = 1·69897; then log. $50+x\times$ log. 1·06 = 2·43296. Again, log. $(100+6x)$ = log. 274 = 2·43775; ∴ an excess of ·00479 remains on the right-hand side of the equation, by taking the difference. Now, put $x=29\cdot5$; then log. $50+x$ log. 1·06 = 2·445615, and log. $(100+6x)$ = log. 277 = 2·44248; ∴ an excess of ·003135 remains on the left-hand side of the equation; then, by proceeding as in Double Positions, we shall find 29·3022 = 29 years, 3mo. 18·792 days.

This question was solved in the same manner by Mr. Maccully.

QUESTION III.—*By Tyro, Lexington, Kentucky.*

To find the values of x, y, z, from the equations $z(x+y+z)=6$, $y(x+y+z)=12$, $x(x+y+z)=18$.

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FIRST SOLUTION.—*By Mr. Henry Darnall, Philadelphia.*

Add the three given equations together, and the square root of the sum will be $x+y+z=6$. Then divide each of the three given equations by this last equation, and we shall obtain respectively $x=3$, $y=2$, and $z=1$.

This question was solved in the same manner by Messrs. Knight, Alsop, Evans, Mathetus, and Sweeny.

SECOND SOLUTION.—*By Mr. Wm. Vodges, Edgemont, Del. Co. Pennsylvania.*

Multiplying the first equation by y, and the second by z, we have

$$yz(x+y+z)=6y,$$
$$yz(x+y+z)=12z;$$

$\therefore$ by Subtraction, $0=12z-6y$, or $y=2z$.

Again, multiplying the first equation by x, and the second by z, we have

$$xz(x+y+z)=6x,$$
$$xz(x+y+z)=18z;$$

$\therefore$ by subtraction, $0=18z-6x$, or $x=3z$.

Substitute the above values of x and y in the first equation, and we have by reduction, $6z^2=6$; $\therefore$ $z=\pm 1$, $y=\pm 2$, and $x=\pm 3$. Hence, taking the positive values, the numbers are 1, 2, and 3.

THIRD SOLUTION.—*By Mary Bond, Fredericktown, Maryland.*

Dividing the 2d and 3d equations by the first, we have $\frac{x}{z}=3$, and $\frac{y}{z}=2$, or $x=3z$, and $y=2z$:

hence, $x+y+z=6z$, and $z(x+y+z)=6z^2=6$;
$\therefore$ $z=1$, consequently $y=2$ and $x=3$.

This question was solved in a similar manner by Messrs. J. Ingersoll Bowditch, Kells, and Hamilton.

FOURTH SOLUTION.—*By Mr. John Swinburne, Brooklyn, L. I.*

It is evident that $x=3z$, and $y=2z$; which values of x and y substituted in any one of the given equations, we obtain $z=1$, consequently $x=3$, and $y=2$.

QUESTION IV.—*By Mr. Michael Floy, New-York.*

There are three numbers in harmonical proportion, the sum of the third is 18, and the product of the three numbers is 576.

Quere, the numbers.

FIRST SOLUTION.—*By Mr. Edward Giddins, Youngstown, Fort Niagara, State of N. York.*

Let x=the first number, then by the question $18-x=$ third number, and by harmonics $\frac{18x-x^2}{9}$ will be the second number, and by the second condition of the question $x\times\frac{18x-x^2}{9}\times\overline{18-x}=576$, that is, $\frac{324x^2-36x^3+x^4}{9}=576$, and this reduced gives $x^4-36x^3+324x^2=5184$, or extracting the square root of both sides we have $x^2-18x=\pm 72$, whence $x=6$ or 12, for the first number, $\frac{18x-x^2}{9}=8$, for the second number, and $18-x=12$ or 6, for the third number, the numbers required are therefore 6, 8, and 12.

The Solutions of Messrs. Kells and Hamilton were similar to this.

SECOND SOLUTION.—*By Mr. John F. James, Trenton.*

Let x=first term, y=second, and $18-x$=third; put $a=18$, $b=576$. Then by the ques. $x:a-x::y-x:a-x-y$ and $x\times y\times(a-x)=b$, which gives $x^2-ax=-\frac{ay}{2}$, and $ax-x^2=\frac{b}{y}$, by adding these together, we get $\frac{b}{y}=\frac{ay}{2}$, that is, $y=\sqrt{\frac{2b}{a}}=8$. Again, $x^2-ax=-\frac{ay}{2}$, completing the square, extracting the root, &c. $x=9\pm 3$; whence the progression is 6, 8, 12.

THIRD SOLUTION.—*By Mr. James Divver, South Carolina, Col. Columbia.*

Let x, y and z be the three numbers, then will $x+z=18$ and $xyz=576$, and by harmonical proportion $x:z::x-y:y-z$. Hence, $xy-xz=zx-zy$ or $xy+zy=2zxy$ or $y=\frac{2zx}{z+x}$. From 2d equation $y=\frac{576}{xz}$. Substitute 18 for $z+x$ and it will be $\frac{2zx}{18}=\frac{576}{zx}$ or $2z^2x^2=10368$ or $z^2x^2=$

5184. Hence $zx=72$, or $x=\frac{72}{z}=18-z$, which gives $z^2-18z^2=-72$; whence $z=9\pm\sqrt{18-72}=9\pm3=6$ ∴ The numbers are 12, 8, and 6.

Fourth Solution.—*By Tyro, Lexington, Kentucky.*

Let x, y and z represent the numbers; then, $x+z=18$, $xyz=576$, and $x:z::x-y:y-z$; whence $2xz=(x+z)y=18y$, or $xz=9y$; ∴ $9y^2=576$, or $y=8$. Consequently $xz=72$, and $x+\frac{72}{x}=18$, or $x^2-18x=72$;

∴ $x=12$, and $z=6$.

Fifth Solution.—*By Mr. James Farrell, New-York.*

Let x, y, z, represent the required numbers, then by harmonic proportion $x:z::x-y:y-z$, from which $y=\frac{2zx}{18}$, because $x+z=18$. Again, $x\left(\frac{2zx}{18}\right)z=\frac{2x^2z^2}{18}=576$ per question, cleared is $2x^2z^2=10368$, dividing by 2 and extracting the root $xz=\sqrt{5184}=72$. Now there is given the sum and product of two numbers to find them, viz. $x+z=18$, and $xz=72$; now $x=18-z=\frac{72}{z}$, whence $18z-z^2=72$, from which Quadratic $z=\pm12$ or 6, and consequently the numbers are 12, 8 and 6, as required.

Question V.—*By Tyro, Lexington, Kentucky.*

A person is entitled to receive an annuity of a certain sum for ever: now the ratio of the present value of each payment to that preceding, is equal to the first payment, and it is required to determine the value of each payment when due, and the present value of the perpetuity, allowing discount at $\frac{5}{21}$ per cent. of the present value of the perpetuity.

First Solution.—*By Tyro, the Proposer.*

Assume x = each annual payment, as also the ratio of

the payments, and y = the present value of the perpetuity, then $x+x^2+x^3+x^4+$ &c. $=y$. Assume $x=Ay+By^2+Cy^3+$&c.

then $x=Ay+By^2+Cy^3+Dy^4+$&c.
$x^2=A^2y^2+2ABy^3+B^2y^4+$&c.
$+2ACy^4+$&c. $\Big\}=y$, whence
$x^3=A^3y^3+3A^2By^4+$&c.
$x^4=A^4y^4+$&c.

$A=1$, $B=-1$, $C=1$, $D=-1$, &c. and therefore we have $x=y-y^2+y^3-y^4+$&c. $=S$, by assumption; then $-y^2+y^3-y^4+$ &c. $=S-y$. Subtracting this from the preceding, leaves $y+y^2-(y^2+y^3)+(y^3+y^4)-$&c. $=y$, which becomes $1+y.(y-y^2+y^3-y^4+$&c.$)=y$, therefore $y-y^2+y^3-y^4+$&c. $=x=\frac{y}{1+y}$, and this being also the ratio of the present value of the terms, or each term being the $\frac{y}{1+y}$ part of the preceding, therefore $\frac{1}{1+y}$ is the discount, but $\frac{1}{1+y}$ is the same part of $\frac{y}{1+y}$ that $\frac{100}{y}$ is of 100, or $\frac{1}{1+y}$ is the $\frac{1}{y}$ part of $\frac{y}{1+y}$; therefore $\frac{100}{y}$ expresses the discount per cent.; then by the question $\frac{100}{y} : y :: 9 : 25$, whence $y^2=277{\cdot}77$ and $y=16{\cdot}66$ and $\frac{100}{16\frac{2}{3}}=6$ per cent. the discount, also $x=\frac{16\frac{2}{3}}{17\frac{2}{3}}=\frac{100}{106}$.

Second Solution.—*By Mr. Benjamin Hallowell, Alexandria, D. C.*

Let x = the present value of the first payment which I consider due one year hence, then the first payment or the annuity must be the amount of x for 1 year at $\frac{5}{24}$ per cent. $=\frac{2509x}{2500}=ax$, putting $a=\frac{2509}{2500}$ the amount of 1 pound

or 1 dollar, for one year. Now, by the question, the present values of the several payments are $x, \frac{1}{a}, \frac{1}{a^2x}, \frac{1}{a^3x^2}, \frac{1}{a^4x^3}$, &c. *ad infinitum*, the sum of which series is $\frac{ax^2}{ax-1}$ =the present value of the whole perpetuity. But by Wood's Algebra, Art. 404, $\frac{ax}{a-1}$ = its present value. Hence $\frac{ax^2}{ax-1}=\frac{ax}{a-1}$; from which $x = 1$, and $\frac{ax}{a-1} = \frac{2509}{9}=277\frac{7}{9}$ the present value of the whole perpetuity. The value of each payment or the annuity $=ax=\frac{2509}{2500}$.

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THIRD SOLUTION.—*By Nemo, New-York.*

Let x=first payment, then $x, 1, \frac{1}{x}, \frac{1}{x^2}$, &c. will be the payments. $S=\frac{x}{x-1}+x$, S=the sum of the series. And, as I suppose, the annuity given=(a), we have this equation

$$\frac{9}{25}\left(\frac{x}{x-1}+x\right) : 100 :: a : \frac{x}{x-1}+x,$$

from which all is known.

—

FOURTH SOLUTION—*By John Capp, Esq. Harrisburg, Penn.*

Suppose the annuity or sum certain to be payable annually, and let r denote $1 increased with its interest for a year, or the amount of $1 in a year: then $\frac{1}{r}$ will be the present value of $1 to be received a year hence; also $\frac{1}{r^2}$ will be the present value of $1 to be received at the end of two years: and $\frac{1}{r^3}$ will be the present value of $1 to be received at the end of three years, &c. And the

present value of the perpetuity, whose first term is $\frac{1}{r}$, the series $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}+$ &c. continued *in infinitum.*

But we know that the sum of this series infinitely continued is $S=\frac{1}{r-1}$, (and when the annuity is a, then $S=\frac{a}{r-1}$;) now the ratio of this series of present values is r, consequently the first is also $=r$, and of course each payment when due, will be $=r$. But since the first term of the perpetuity whose sum is $S=\frac{1}{r-1}$, begins with $\frac{1}{r}$, and the sum of the series required being $r+1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$, &c.; we must therefore add the two terms $r+1$, to the sum of the series above found, and we get the sum of the whole $=\frac{1}{r-1}+r+1=\frac{r^2}{r-1}$. Now, by taking $r=1{\cdot}05$, $r-1={\cdot}05$, and $r^2=1{\cdot}1025$, we thence obtain $\frac{r^2}{r-1}=22{\cdot}05$=the perpetuity, and $100:\frac{5}{95}::22{\cdot}05:7{\cdot}938\div 100={\cdot}07938$; hence, $22.05-{\cdot}07938=21{\cdot}97062$ dollars.

[It appears, from several ingenious communications, that this question was very obscurely enunciated, and, consequently, that it must have been unintelligible to many of the contributors. EDITOR.]

QUESTION VI.—*By John Capp, Esqr. Harrisburg, Penn.*

A and B purchased a valuable farm, containing 900 acres of land, at the rate of $2 per acre, which they paid equally between them; but on dividing the same, A got that part of the farm which contained the best of the improvements, and agreed to pay 45 cents per acre more than B, how many acres had each, and at what price?

FIRST SOLUTION.—*By Mr. H. Crosby, Hyattstown, Maryland.*

Let $a=900$

and x=the number of acres A had,
$a-x$=the number of acres B had,

$\frac{a}{x}$=the price per acre paid by A,

$\frac{a}{a-x}+\frac{45}{100}=\frac{a}{a-x}+\frac{9}{20}$ the price per acre paid by B;

then $\frac{a}{x}=\frac{a}{a-x}+\frac{9}{20}$

cleared of fractions $20a^2-200x=20ax+9ax-9x^2$.

Transposing and uniting terms $9x^2-49ax=20a^2$,

Dividing by 9 $x^2-\frac{49ax}{9}=\frac{20a^2}{9}$

Completing square $x^2-\frac{49ax}{9}+\frac{2401a^2}{324}=\frac{2401a^2}{324}-\frac{20a^2}{9}=\frac{1681a^2}{324}$.

Extracting root $x-\frac{49a}{18}=\frac{a}{18}\sqrt{1681}=\frac{a}{18}\times 41$,

which in numbers is $x-\frac{44100}{18}=\frac{36900}{18}$

or $x-2490=2090$,

whence $x=2490-2090=400$ A's share,

and $900-400=500$ B's share,

$\frac{900}{400}=\$2.25$ price per acre paid by A,

and $\frac{900}{500}=\$1.80$ price per acre paid by B.

45 cents difference.

SECOND SOLUTION.—*By Mr. Gerardus B. Docharty, Flushing, Long Island.*

Let x=the acres that belonged to A, then $900-x$=those of B's; also let y=the cents per acre given by B, and $y+45$=A's, then per query $xy+45x=900y-xy$, and $xy+45x+900y-xy=180000$; from the first of these equations $x=\frac{900y}{2y+45}$, and from the 2d. $x=4000-20y$; put

these two values of x =, then $4000-20y=\frac{900y}{2y+45}$, hence by clearing of fractions, reduction &c. $y^2-155y=4500$, which, resolved by quadratics; gives. $y=180$ cents the price of an acre of B's land, hence $180+45=\$2{\cdot}25=$the price of an acre of A's, whence $x=4000-\frac{20\times180}{}=400$ the number of acres belonging to A, and $900-400=500$ $=B$'s. Consequently A had 400 acres at $\$2{\cdot}25$ per acre, and B had 500, at $\$1{\cdot}80$ per acre.

This question was solved in a similar manner by Messrs. Sweeny, Hamilton, and Wright.

THIRD SOLUTION.—*By Mr. Wm. Vodges, Edgemont, Del. Co. Penn.*

Let x=the number of acres bought by A, and y=the number of acres bought by B. $\frac{90000}{x}$= price of A's land per acre in cents, and $\frac{90000}{y}$ = price of B's ditto.

Then by the question we have $x+y=900$, or $x=900-y$, and $\frac{90000}{x}=\frac{90000}{y}+45$, from which $x=\frac{2000y}{2000+y}$, whence $\frac{2000y}{2000+y}=900-y$, from which we get $y^2+3100y=1800000$, from this equation we obtain $y=500$ acres bought by B, and $x=400$ acres bought by A. The price of A's land per acre is $\$2{\cdot}25$, and B's do. $\$1{\cdot}80$.

Mathetus and Mary Bond's solutions are similar to this.

QUESTION VII.—*By Mr. Henry Darnall, Philadelphia.*

Ye sons of science tell me when,
Between the hours of twelve and one,
My watch's hands themselves will place,
To form an angle of such space,
If its cotangent you prepare,
And of its secant get the square,
Their product shall in numbers come,
To what you style a minimum?

FIRST SOLUTION.—*By Mr. Solomon Wright, Buck's Co. Penn.*

Let x=the tangent and y=the secant of the angle that the hands make with each other, and r=the radius,

then $y^2-x^2=r^2$; and as $x:r::r:\frac{r^2}{x}$=cotangent; therefore $\frac{r^2y^2}{x}$= minimum.

$\therefore \frac{2r^2xy\dot{y}-r^2y^2\dot{x}}{x^2}=0$, or $2r^2xy\dot{y}-r^2y^2\dot{x}=0$, and $\dot{y}=\frac{r^2y^2\dot{x}}{2r^2xy}$;

again, from the first equation $2y\dot{y}-2x\dot{x}=0$, or $\dot{y}=\frac{x\dot{x}}{y}$.

$\therefore \frac{r^2y^2\dot{x}}{2r^2xy}=\frac{x\dot{x}}{y}$, and by clearing of fractions we have $y^2\dot{x}=2x^2\dot{x}$ and $y^2=2x^2$, then by substituting this value in the first equation, we have $2x^2-x^2=r^2$ or $x^2=r^2$ or $x=r$; therefore, it is obvious that the angle that the hands must make with each other is=45°, or $7\frac{1}{2}$ minutes of time. Now put x=the time past 12, then by the question $x-\frac{x}{12}=7{\cdot}5$ minutes, or $x=\frac{90}{11}$=8m $10\frac{10}{11}$ sec. past 12, the time required.

SECOND SOLUTION.—*By Professor Divver, S. C. College, Columbia.*

Put x=cosine of the arc, then will $\sqrt{1-x^2}$ be the sine, supposing the radius unity. But the sine will be to the cosine as rad. to the cotan. that is, $\sqrt{1-x^2}:x::1:$ cotan. $=\frac{x}{\sqrt{1-x^2}}$. Also $x:1::1:$ sec. Therefore the secant of the arc will be $=\frac{1}{x}$. The square of it is $=\frac{1}{x^2}$, which multiplied by $\frac{x}{\sqrt{1-x}}$ gives $\frac{x}{x^2\sqrt{1-x^2}}=\frac{1}{x\sqrt{1-x^2}}$ $=\frac{1}{\sqrt{x^2-x}}$. The fluxion of which $=-x\dot{x}+2x^3\dot{x}=0$,

or $2x^2=1$

$x=\sqrt{\frac{1}{2}}$.

The sine will also be $\sqrt{\frac{1}{2}}$ and the secant 2. Whence the hands of the watch will form an angle of 45°, and sec. squared and × by cotan. produces 4 = minimum. The time will be found to be $8\frac{2}{11}$ min. past 12 o'clock.

THIRD SOLUTION.— *By Mr. J. Ingersoll Bowditch, Boston.*

Put x for the angle which the hands of the watch make. Hence we have by the question $\cot x \times \sec x^2 =$ a minimum; and since $\cot = \frac{\cos}{\sin}$, and $\sec = \frac{1}{\cos}$, we get $\frac{\cos x}{\sin x} \times \frac{1}{\cos x^2} = \frac{1}{\sin x \cos x} =$ a minimum, or $\sin x \cos x =$ a maximum; taking the differential we have $dx \cos x^2 - dx \sin x^2 = 0 \therefore \cos x = \sin x$; hence the angle must be $45° = 7\frac{1}{2}$ minutes. Hence the question is to find when the hands will be 45° or $7\frac{1}{2}$ minutes apart, which, by adding its $\frac{1}{11}$th to it, gives 8 minutes 11 seconds after 12 o'clock for the time.

FOURTH SOLUTION.—*By Mr. John F. James, Trenton.*

Let $x =$ cot. $y =$ sec. $a =$ rad. $v =$ tan. $u =$ cosec. Then $x^2 = u^2 - a^2$, and $y^2 = a^2 + v^2$, adding these $y^2 + x^2 = u^2 + v^2$. By question $xy^2 =$ minimum, in which x decreases, whilst y increases; hence, by taking the fluxions we have $2xy\dot{y} - y^2\dot{x} = 0$, or $\dot{y} = \frac{y\dot{x}}{2x}$, and $2y\dot{y} - 2x\dot{x} = 0$, or $\dot{y} = \frac{x\dot{x}}{y}$; whence $\frac{y\dot{x}}{2x} = \frac{x\dot{x}}{y}$, or $y^2\dot{x} = 2x^2\dot{x}$, that is, $y^2 = 2x^2$, or $x^2 = \frac{1}{2}y^2 = 45°$. Or, by substituting in the third equation $3x^2 = u^2 + v^2$, or $x^2 = \frac{u^2 + v^2}{3} = 45°$. Whence 330° : 45° :: 60 minutes : $8\frac{2}{11}$ minutes past 12 hours.

FIFTH SOLUTION.—*By Mr. James Maccully, N. Brunswick, N. Jersey.*

Let $x =$ the tan. of the angle,

then $\frac{1}{x} =$ the cotangent,

and $\sqrt{(1+x^2)} =$ secant;

$\therefore \frac{1+x^2}{x}$ = a max. by the question;

and $\frac{2x^2\dot{x}-\dot{x}-x^2\dot{x}}{x^2}=0$, or $x^2=1$;

$\therefore x=1=$ tangent of 45°.

Hence it is 8 minutes, $10\frac{10}{11}$ seconds past 12.

QUESTION VIII.—*By Mr. Wm. Kells, Bergen, N. Jersey.*

The length of the chord of an arc of a circle being given, and the length of a perpendicular let fall on the arc from a given point in the chord, it is required to determine the length of the radius.

SOLUTION—*By Messrs. George M. Alsop, and Edward Evans, Philadelphia.*

It is evident that a straight line drawn perpendicular to the arc of a circle passes through the centre, therefore by Euclid 35, III. we have

$\frac{1}{2}\left(\frac{ab}{c}+c\right)$ = radius; where a and b are the segments of the chord, and c the length of the perpendicular let fall on the arc.

This question was solved in the same manner by Mr. James Hamilton.

QUESTION IX.—*By Mr. Farrell Ward, New-York.*

To find the area and length of a curve whose nature is such that the subtangent is always equal to the rectangle of the ordinate and subnormal.

FIRST SOLUTION.—*By Mr. Elias Lynch, New-York.*

Let x and y be the abscissa and ordinate of the required curve; then by the question we have

$$\frac{ydx}{dy} : \frac{ydy}{dx} :: y : 1$$

$$dx^2 : dy^2 :: y_{\frac{1}{2}} : 1$$

$$dx : dy :: y^{\frac{1}{2}} : 1$$

$$\int dx = \int y^{\frac{1}{2}} dy$$

$x=\frac{2}{3}y^{\frac{3}{2}}$, a case of the cubical parabola.

The area $=\int ydx=\int y^{\frac{3}{2}}dy=\frac{2}{5}y^{\frac{5}{2}}=\frac{3}{5}xy$, which wants no correction. Length $=\int(dy^2+dx^2)^{\frac{1}{2}}$ or $\int dy(1+y)^{\frac{1}{2}}=\frac{2}{3}(1+y)^{\frac{3}{2}}+C$; and when the length $=0$, $y=0$, $C=-\frac{2}{3}$; hence the length $=\frac{2}{3}(1+y)^{\frac{3}{2}}-\frac{2}{3}$.

Mr. Farquhar solved this question in a similar manner.

SECOND SOLUTION.—*By Nemo, New-York.*

Let x and y be the co-ordinates $\frac{dxy}{dy}=\frac{dyy^2}{dx}$ and $A=\frac{2}{5}\times y^{\frac{5}{2}}+c$, and $3=\frac{2}{3}(1+y)^{\frac{3}{2}}+c$, c being the correction.

THIRD SOLUTION.—*By Professor Strong, Hamilton College, State of New-York.*

Let x denote the abscissa and y the ordinate, and a a given quantity instead of 1, as in the question; then is $\frac{ydx}{dy}$ = the subtangent, and its subnormal $=\frac{ydy}{dx}$; $\therefore\frac{y^2dy}{dx}=\frac{aydx}{dy}$, and $dy^2y=a\cdot dx^2$; $\therefore y^{\frac{1}{2}}dy=a^{\frac{1}{2}}dx$; $\therefore\frac{2}{3}y^{\frac{3}{2}}=a^{\frac{1}{2}}x$, and $y^3=a'x^2$, which needs no correction, supposing x and y to commence together: this is the equation to the semicubical parabola, and its length $=(9y+4a')^{\frac{3}{2}}-8a^{\frac{3}{2}}$, the length being reckoned from the commencement of the abscisses and ordinates; and its area reckoning from the same point $=\frac{3}{5}(\frac{y^5}{a'})^{\frac{1}{2}}$, supposing that $\frac{9}{4}a=a'$. If $a=1$, then the results will be according to the conditions of the question.

FOURTH SOLUTION.—*By Mr. J. Ingersoll Bowditch, Boston.*

Put x for the absciss and y for the ordinate of the curve. The subtangent being $\frac{ydx}{dy}$ and the subnormal $\frac{ydy}{dx}$, we

have per question $\frac{ydx}{dy}=\frac{y^2dy}{dx}$ $\therefore ydx^2=y^2dy^2$ $\therefore dx^2=ydy^2$ $\therefore dx=y^{\frac{1}{2}}dy$, taking the integral of this we have $\frac{y^{\frac{3}{2}}}{\frac{3}{2}}+a=x$ $\therefore y^{\frac{3}{2}}+\frac{3}{2}a=\frac{3}{2}x$; suppose x and y to commence together, then we have $x=\frac{2}{3}y^{\frac{3}{2}}$; squaring this gives $x^2=\frac{4}{9}y^3$, *the equation of a cubic parabola.* From $x=\frac{2}{3}y^{\frac{3}{2}}$, we get $dx=y^{\frac{1}{2}}dy$, and as the differential of an area is ydx, we have for the difference of this area $y^{\frac{3}{2}}dy$; hence the integral of this, which is $\frac{2}{5}y^{\frac{5}{2}}$, is the area of this curve; and $\frac{2}{5}y^{\frac{5}{2}}=\frac{2}{5}yy^{\frac{3}{2}}$, substituting in this $x=\frac{2}{5}y^{\frac{3}{2}}$, or $y^{\frac{3}{2}}=\frac{3}{2}x$, gives $\frac{2}{5}y\times\frac{3}{2}x=\frac{3}{5}xy$ *for the area.* The d (of an arc)$=\sqrt{dx^2+dy^2}$, and as we have above $dx^2=ydy^2$, by substituting this in the equation of the d (of an arc), we obtain $\sqrt{ydy^2+dy^2}$, or $dy\sqrt{y+1}$ for the d (of an arc of this curve). Put $\sqrt{y+1}=z$ $\therefore y+1=z^2$, the d of this is $dy=2zdz$. Hence the d (of the arc of this curve)$=2z^2dz$. Integrating this gives $\frac{2}{3}z^3$ or $\frac{2}{3}(y+1)^{\frac{3}{2}}$ *for the length of the arc.*

From $x=\frac{2}{3}y^{\frac{3}{2}}$, we have $dx=y^{\frac{1}{2}}dy$, substituting this in the general equations of subnormal and S. Tangent, we obtain for the S. N. and S. T. of this curve $y^{\frac{1}{2}}$ and $y^{\frac{3}{2}}$, which answers the condition of the question.

QUESTION X.—*By Mr. Edward C. Ward, New-York.*

Writers on navigation say that Middle Latitude Sailing is incorrect, and that Mercator's Sailing is strictly true: now, in sailing from New-York to Gibraltar, whether is it safer to compute the daily difference of longitude by Middle Latitude, or Mercator's Sailing?

FIRST SOLUTION.—*By Professor Strong, Hamilton College, State of N. York.*

By the principles of parallel sailing I derive $\frac{d}{\cos.(M-\phi)}$ $+\frac{d}{\cos.(M+\phi)} = dL+d^nL$, or by reduction - - - - - $\frac{2d \cos. M \cos. \varphi}{\cos.^2 M \cos.^2\varphi - \sin.^2\varphi \sin.^2 M} = dL+d^nL$, in which M denotes the middle latitude, and φ half the difference of latitude, and d denotes the infinitely small departure corresponding to the infinitely equal parts into which the rhumb line passing through the two places may be conceived to be divided; the two equal portions of the rhumb on each side of the middle parallel being conceived to be divided into the same number of infinitely small equal parts, and, as is well known, the infinitely small departure belonging to the equal portions of the rhumb are all equal. Now $\frac{d}{\cos(M-\varphi)}$ denotes the infinitely small portions of the rhumb which I put equal to dL, and also $\frac{d}{\cos.(M+\phi)}$, which I suppose $=d^nL$, denotes the infinitely small difference of longitude belonging to the last infinitely small portion of the rhumb; by the terms first and last, I mean the parts which are nearest and most remote from the equator, and it must be noted that the arcs M and φ are supposed to be taken to radius, (1). Now, from the equation $\frac{2d \cos. M. \cos. \varphi}{\cos.^2 M \cos^2\varphi - \sin.^2 M \sin.^2\varphi} = dL+d^nL$, it appears that if M is much less than 90°, and that φ is a very small arc, we may suppose $\cos.\varphi = 1$ very nearly, and also the term $\sin.^2 M \sin.^2 \varphi$ is so small with respect to $\cos.^2 M \cos.^2\varphi$, that it may be neglected without much error.

Making the aforesaid reductions we have $\frac{2d}{\cos. M} = dL + d^nL$ very nearly, and it is manifest that the smaller φ is, as well as M, the more correct is the result, so that if $\varphi = 0$, it is exactly true. For example, suppose $M = 60°$, and $\varphi = 5°$, then is $\cos.^2\varphi \times \cos.^2 M = 0{\cdot}24810$, and $\sin.^2 M \times \sin.^2\varphi = 0{\cdot}0056971 =$

0·0056971, so that $\cos.^2\varphi \cos.^2 M$ is about 43 times the value of $\sin.^2 M \sin.^2\varphi$; hence the error committed even when the middle latitude is 60° and the difference of latitude 10°, is very small. By the same process of reasoning, I derive from the first and last portions of the rhumb but one, the equation $\frac{2d}{\cos. M} = d^1L + d^{n-1}L$, and for the two next portions, I have $\frac{2d}{\cos. M} = d^2L + d^{n-2}L$, and so on; hence by addition I have the equation $\frac{2d}{\cos. M} + \frac{2d}{\cos. M} + \&c. = dL + d^nL + d^1L + d^{n-1}L + \&c.$; but $\frac{2d}{\cos. M} + \frac{2d}{\cos. M} + \&c. = \frac{D}{\cos. M}$, and $dL + d^nL + d'L + d^{n-1}L + \&c. = L \therefore \frac{D}{\cos. M} = L$, in which D denotes the whole departure, and L denotes the whole difference of longitude made. This is the known theorem in middle latitude sailing. From what has been done, it is manifest that the smaller the difference of latitude, if the middle latitude does not much exceed 45°, that the more correct is the result. Now, in Mercator's method, as the tables of meridional parts are seldom carried into decimals, it is manifest that the smaller the difference of latitude the more inaccurate the result; so that it is very probable as middle latitude converges towards the truth, when Mercator's varies from it, that when the difference of latitude is very small, middle latitude will give a more correct result than Mercator's method; but by computing the tables of meridional parts to a sufficient number of decimals, the difference of longitude may be obtained to any degree of exactness required, and that whether the earth be considered as a sphere or spheroid. Now as M in the question is about 38° 23′ and $2\varphi = 4°\ 37'$ very nearly, it is manifest from what has been done, that the result by middle latitude will be very accurate, and probably more correct than the common tables of meridional parts will give.

SECOND SOLUTION.—*By Dr. Bowditch, Boston.*

The method by Mercator's sailing is in theory ab-

solutely correct, but in practice this is not the case, because the meridional difference of latitude is generally given only to miles, and when places are situated nearly in the same parallel of latitude, but on distant meridians, the error of one mile in the meridional parts would cause a greater error in the course or distance than would be committed in middle latitude sailing. The consequence is that almost 999 cases out of 1000, are by seamen solved by middle latitude sailing. The finding how much the difference, is a question of no difficulty. The best result is *practically* found in a correct manner by all seamen.

THIRD SOLUTION.—*By Nemo, New-York.*

In answer to this question, I agree with what our authors say, that mid. lat. is false and Mercator's right, the earth supposed to be a perfect sphere; but should we take into account the true fig. of the earth, it would make a difference; but I think the earth in the question is supposed a perfect sphere, and therefore say no more on the question.

FOURTH SOLUTION.—*By Mr. Charles Farquhar, Alexandria, D. C.*

If the earth be considered a sphere, then the principles of Mercator's sailing being rigidly true, and those of middle latitude sailing only an approximation, it would therefore be more incorrect to make use of the former method, in determining the difference of longitude. In the present instance the course being nearly east, the error arising from the latter method would be very inconsiderable; but the general result must, it appears to me, still involve an error. If the earth is considered a spheriod, then, as was satisfactorily shown in No. 2 of the Diary, the fundamental principles of plane sailing, (and consequently of Mercator's,) are incorrect, and the question, in this view, would require more particular data than is given.

QUESTION XI.—*By Mr. Martin Roche, Philadelphia.*

It is required to determine the place of a planet in its

elliptical orbit about the sun is one of the foci, when the angular velocity of the planet about the other focus is a maximum or minimum.

FIRST SOLUTION.—*By Dr. Bowditch, Boston.*

Let the planet's distance from the sun be r, its anomaly counted from the perihelion v, and the distance and corresponding angle counted from the other or upper focus be R and V, the mean distance being a, so that $R+v=2a$. The radii R, r (by a well known property of the ellipsis) make the same angle A, with the tangent to the curve, (or the direction of the planet's motion), so that if the planet describe in the direction of the tangent an infinitely small space ds, this reduced to the direction perpendicular to the radius r or R, will be ds sin. A; consequently the angle subtended by radii drawn from the sun to the extremes of the space ds will be $\frac{ds.\ \text{sin.}\ A}{r}$, and the angle formed at the other or upper focus $\frac{ds.\ \text{sin.}\ A}{R}$. These quantities are to each other as $\frac{r}{R}$ to 1. Now the regular velocity of the planet about the sun being as $\frac{1}{r^2}$, the angular velocity about the other focus will be as $\frac{r}{R}\times\frac{1}{r^2}$; or as $\frac{1}{Rr}$. Its differential put equal to 0 gives $Rdr+rdR=0$, but $R+r=2a$ gives $dR+dr=0$, substituting this value of $dR=-dr$ in the former equation gives $Rdr-rdr=0$, which is satisfied by putting $dr=0$ or $R-r=0$. Now $dr=0$ at the extremes of the transverse axis where $R=a\pm e$, $r=a\mp e$, $Rr=aa-ee$ (e being half the local distance) and the other equation $R-r=0$ combined with $R+r=2a$ gives $R=r=a$, and $Rr=aa$; hence it is evident that the *least* angular velocity about the upper focus is when the planet is at the extremes of the conjugate axis where $\frac{1}{Rr}=\frac{1}{aa}$, and the *greatest* velocity at the extremes of the tranverse axis where $\frac{1}{Rr}=\frac{1}{aa-ee}$.

SECOND SOLUTION.—*By Professor Strong, Hamilton College, State of New-York.*

Let x denote the distance of the planet from the sun, and let $d\phi$, denote its angular motion in a given time; then by Kepler's law of the equal description of areas, we have $x^2 d\varphi = c = a$ constant $\therefore x d\varphi = \frac{c}{x}$ but $\frac{x d\phi}{a-x}$ equals the angular motion round the other focus in the same given time, supposing that a denotes the transverse diameter; this is derived from the circumstance, that x and $a-x$, make equal angles with the curve. But $\frac{x d\varphi}{a-x} = \frac{c}{ax-x^2} =$ min. or max. $\therefore$ $ax-x^2 =$ max. or min. which gives $x = \frac{a}{2}$ $\therefore$ the angular motion is a minimum at the extremities of the conjugate, and it is a maximum at the the extremities of the transverse, because that $dx=0$ at those points.

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THIRD SOLUTION.—*By Mr. Dubre Knight, Alex. D. C.*

Let r = planet's distance from the sun in one focus,
r' = do. from the other focus,
v = its velocity perp. to the line represented by r,
v' = do. do. by r'.

Then, from the nature of the ellipse we have $v=v'$. Now the angular velocity about the centre of the circle whose rad. is r' varies as $\frac{v'}{r'}$, that is, as $\frac{v}{r'}$; but v varies as $\frac{1}{r}$ $\therefore$ $\frac{v'}{r'}$ varies as $\frac{1}{rr'}$, and $r+r'$ being constant, the angular velocity will be a maximum at the extremities of the transverse, and a minimum at the extremities of the conjugate axis.

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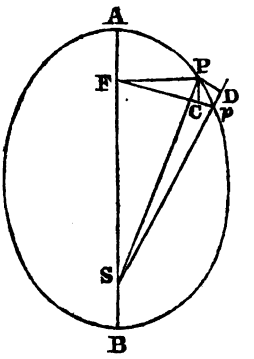
FOURTH SOLUTION.—*By Mr. Benjamin Hallowell, Alexandria, D. C.*

Let APB represent the orbit of a planet, S and F the foci, the sun being at S, P and p, two situations of the planets indefinitely near each other. Join FP, Fp, SP and

Sp, and on Fp and Sp produced let fall the perpendiculars PC and PD. Now since pP is indefinitely small, it may be considered as a tangent at p. Hence by conic sections the angles PpD and PpC, and consequently PD and PC are equal.



Also, since the area of PSp is constant for the the same time, PD or PC varies as $\frac{1}{Sp}$; but the angle PFp varies as $\frac{PC}{Fp}$, which varies as $\frac{1}{Fp \times Sp}$ $\left(\text{since } PC \text{ varies as } \frac{1}{Sp}\right)$ or inversely as $Fp \times Sp$, the sum of which being constant, the rectangle will be the greatest and consequently the angular velocity about F the least when $Fp = Sp$, or when the planet is at its mean distance from the sun; and this rectangle will be the least, and consequently the angular velocity about F the greatest, when the planet is in its aphelion or perihelion at which places it is equal.

QUESTION XII.—*By Tyro, Lexington, Kentucky.*

In the collision of unelastic bodies the relative velocity divided into the ratio of the masses, gives the velocities lost and gained, and consequently the velocities after impact, that is, the part proportional to the greater mass, is the velocity lost or gained by the lesser mass; and if we divide twice the relative velocity in the same manner, we obtain similar results for the collision of elastic bodies: the demonstration is required.

FIRST SOLUTION.—*By Dr. Bowditch, Boston.*

Let M and M' be the bodies, V, V' their velocities before impact, v their common velocity after impact, supposing them non-elastic. V, V', their velocities, if elas-

tic. Then by the usual rules for non-elastic bodies, we have $v=\frac{MV+M'V'}{M+M'}$, whence we get the relative velocities $V-v=\frac{M'}{M+M'}\cdot(V-V')$; $v-V'=\frac{M}{M+M'}\cdot(V-V')$, being the same as was mentioned in the first part of the problem. Now the principles of elasticity explained by various authors requires that the momentum lost by M in consequence of the elasticity should be measured by the same quantity as was lost in the impact of the bodies supposed unelastic, the quantity $V-v$ will then be doubled, and the loss of M will be $\frac{M}{M+M'}\cdot(2V-2V')$: in like manner, the gain of M will be $\frac{M}{M+M'}(2V-2V')$.

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SECOND SOLUTION.—*By Nemo, New-York.*

Elastic bodies restore themselves with the same force by which they are compressed, consequently their velocities will be double non-elastic bodies, therefore, the divisor being the same, we must double the dividend, as the question says.

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THIRD SOLUTION.—*By Mr. Charles Farquhar, Alex. D. C.*

Let A and B represent the elastic bodies, r their relative velocity. Then, (Mechanics,) $A+B : 2A :: r :$ the velocity gained by B in the direction of A's motion, $=2r\times\frac{A}{A+B}$; where it is evident $2r$ is divided in the ratio of A and B; the result is therefore the same as in non-elastic bodies.

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FOURTH SOLUTION.—*By Mr. Benjamin Hallowell, Alex. D. C.*

Let A and B represent the two hard bodies, and v their relative velocities; then Wood's Mechanics, Prop. 44, $A+B : A :: v :$ the velocity gained by A. Also, keeping the same notation for elastic bodies we have by Prop. 46,

$A+B : B : : 2v$: the velocity gained by B; and $A+B : B : : 2v$, the velocity lost by A; from which it appears that this question is answered in the 41 and 45 Propositions of Wood's Mechanics.

QUESTION XIII.—*By Diarius New-York.*

Required the figure of a rectangular parallelopipedon, such that the sum of the edges, the sum of the faces, as well as each edge and each face, may be expressed by numbers which are rational squares.

FIRST SOLUTION.—*By John Capp, Esq. Harrisburg, Penn.*

Supposing the figure of the parallelopiped to be of three dimensions; namely, length, breadth, and depth; and let the longest edges be denoted by x^2, those of the middle length or next longest by y^2, and the shortest by z^2. Then will $4x^2+4y^2+4z^2$ = the sum of all the edges; and $2x^2y^2+2x^2z^2+2y^2z^2$ = the sum of all the faces of the parallelopiped; which formulas must both be squares.

Dividing our first formula by 4, it becomes $x^2+y^2+z^2 = \square = (x+y-z)^2 = x^2+y^2+z^2+2xy-2xz-2yz$; removing equals, dividing by 2, there remains $xz+yz=xy$, whence, $z = \frac{xy}{x+y}$.

By substituting this value of z in our remaining formula, $2\times(x^2y^2+x^2z^2+y^2z^2)$, it will become $2x^2y^2+2.(x^2+y^2) \cdot (\frac{xy}{x+y})^2$. Multiplying the last formula by $(\frac{x+y}{xy})^2$, and reducing, our formula becomes $4x^2+4xy+4y^2$; or dividing by 4, it becomes x^2+xy+y^2; which must also be rendered a square. Let us therefore assume $x^2+xy+y^2=(rx-y)^2=r^2x^2-2rxy+y^2$. Removing equals, and transposing, dividing by x, we have $(r^2-1) \cdot x=(2r+1) \cdot y$; whence, $\frac{x}{y}=\frac{2r+1}{r^2-1}$. Now, here r may be taken at pleasure provided it be assumed greater than 1.

By taking $r=2$, we obtain $x=5$, $y=3$, and $z=\frac{15}{8}$

Multiplying these numbers by 16. we obtain the integers $x=80$, $y=48$, and $z=30$.

We have $(x^2+y^2+z^2).4=9604\times4=(196)^2$, the sum of all the edges; and $(x^2y^2+x^2z^2+y^2z^2).4=45158400=(6720)^2$, the sum of all the faces.

SECOND SOLUTION.—*By Mr. Charles Wilder, Baltimore.*

Put x^2, y^2, and z^2 for the three edges, then by the question $x^2+y^2+z^2=\square$, and $2x^2y^2+2x^2z^2+2y^2z^2=\square=4m^2$, (by putting $2m$ for its side); this gives $y^2+z^2=\frac{y^2z^2}{x^2}(2m^2-1)$; hence we have by the first equation $\frac{x^4+y^2z^2(2m^2-1)}{x^2}=\square$, or $x^4+y^2z^2(2m^2-1)=\square$; which becomes by putting $x^4=(\frac{2-a^2}{2a})^2m^2y^2z^2$, $y^2z^2(\frac{2+a^2}{2a})^2m^2-y^2z^2=\square$, or $(\frac{2+a^2}{2a})^2m^2-1=\square$; put the side of this square equal to $b-(\frac{2+a^2}{2a})m$ and $m=\frac{a(b^2+1)}{2+a^2}$, but $m=\frac{2ax^2}{(2-a^2)yz}$; hence $y=\frac{2x^2(2+a^2)}{z(2-a)(b^2+1)}$; where a, b, x and z, are arbitrary.

THIRD SOLUTION.—*By Mr. Dubre Knight, Alex. D. C.*

Let x^2 y^2 and z^2 denote the three edges about one of the solid angles, then will the conditions be answered by making squares of the two expressions $x^2+y^2+z^2$, and $2x^2y^2+2x^2z^2+2y^2z^2$. Put the 1st$=(x+y-z)^2$, then $x=\frac{yz}{y-z}$, and substituting this in the 2nd, we have $\frac{4}{(y-z)^2}\times(y^2-yz+z^2)$ to make a square, which will be done by putting $y^2-yz+z^2=(y-nz)^2$; whence $y=\frac{n^2-1}{2n-1}z$.

FOURTH SOLUTION.—*By Mr. Charles Farquhar, Alexandria, D. C.*

Let the dimension be x, y, and z. Then,

$\left.\begin{matrix} 4x+4y+4z \\ 2xy+2xz+2yz \\ x,\ y,\ \text{and}\ z \\ xy,\ xz,\ \text{and}\ yz \end{matrix}\right\} = \square^{s.}$ Therefore we have to make, $\left.\begin{matrix} m^2+n^2+r^2 \\ \text{and}\ 2(m^2n^2+m^2r^2+n^2r^2) \end{matrix}\right\} = \square^{s.}$

Assume, $m^2+n^2+r^2 = \overline{m+n-r}^2$, and we get $mn = mr+nr$, (A) by squaring this, and substituting in the other expression, we have, $4(m^2n^2-mnr^2) = \square$, or, $m^2n^2-mnr^2 = \square = \overline{mn-pr}^2$, from which, $r = \frac{2pmn}{p^2+mn} =$ (from A) $\frac{mn}{m+n}$, or, $m = \frac{p^2-2pn}{2p-n}$, where p and n may be assumed at pleasure. If $p = 3$ and $n = 1$, then $m = \frac{3}{5}$ and $r = \frac{3}{8}$, therefore the dimensions are $1, \frac{9}{24}$, and $\frac{9}{64}$, or, in whole numbers, $\overline{24}^2$, $\overline{40}^2$, and $\overline{15}^2$.

FIFTH SOLUTION.—*By Mr. Benjamin Hallowell, Alexandria, D. C.*

Let, y^2, x^2, and z^2 denote the length, breadth, and thickness, respectively; then it is evident that each edge and each face will be a square; hence we have only to make the sum of the edges $4x^2+4y^2+4z^2$, or $x^2+y^2+z^2$ a square, and the sum of the faces $2x^2y^2+2x^2z^2+2y^2z^2$ = a square. To do this, put $x^2+y^2+z^2 = (x+y-z)^2$, and we get $z = \frac{xy}{x-y}$, which being substituted for z in the second, we obtain $2x^2y^2+\frac{2x^4y^2}{(x+y)^2}+\frac{2x^2y^4}{(x+y)^2} = \square$, or, by reducing and throwing out the squares common to each term, $x^2+xy+y^2 = \square$, which put $= (x-\frac{m}{n}y)^2$, and we get $x = \frac{m^2-n^2}{n^2-2mn}y$. Hence, $y = n^2+2nm$, and $x = m^2-n^2$, when m and n may be any numbers of which m is the greater. If we take $m = 2$ and $n = 1$, we get $y^2 = 25$, $x^2 = 9$, and $z^2 = \frac{225}{64}$. To obtain integers, multiply each of these by 64; then $y^2 = 1600$, $x^2 = 576$, and $y^2 = 225$.

Professor Strong's method of solution is nearly similar to this, and his results exactly the same.

QUESTION XIV.—*By Professor Strong, Hamilton College, State of New-York.*

Integrate the equation $\frac{d\varphi}{1 \pm n \sin.^2\phi}$.

—

FIRST SOLUTION.—*By Dr. Bowditch, Boston.*

Instead of $\sin.^2\varphi$, put its value $\frac{1}{2} - \frac{1}{2}\cos.2\varphi$, and the proposed equation becomes $\int \frac{d\varphi}{1 \pm \frac{1}{2}n \mp \frac{1}{2}n\cos 2\varphi}$, and if we put $2\varphi = z$, $2 \pm n = a$, $\mp n = b$, it becomes $\int \frac{dz}{a + b\cos.z}$, which is the equation treated of by La Croix in T. 2. page 103, edit. of his Calcul. Diff. et Integ. His solutions are under various forms, as for example,

when $a < b$, $\int \frac{dz}{a+b\cos.z} = \frac{1}{\sqrt{b^2-a^2}}$ hyp. log. $\left\{ \frac{\sqrt{(b+a)(1+\cos.z)} + \sqrt{(b-a)(1-\cos.z}}{\sqrt{(b+a)(1+\cos.z)} - \sqrt{(b-a)(1-\cos.z}} \right\}$ + const.

$= \frac{1}{\sqrt{b^2-a^2}}$ hyp. log. $\left\{ \frac{1 - \sqrt{\frac{b-a}{b+a}}\tan.\frac{\phi}{}}{1 - \sqrt{\frac{b-a}{b+a}}\tan.\frac{\phi}{}} \right\}$ + const.

when $b > a$ $\int \frac{dz}{a+b\cos z} = \frac{2}{\sqrt{(a^2-b^2)}}$. arc (tan. $= \frac{\sqrt{(a-b)} \cdot \sqrt{(1-\cos.z)}}{\sqrt{(a+b)} \cdot \sqrt{(1+\cos.z)}}$) + const. $= \frac{2}{\sqrt{aa-b^2}}$ arc. $\left(\tan = \sqrt{\frac{a-b}{a+b}}\tan.\frac{\phi}{}\right)$ + const. The deductions from La Croix's values in z being easily made to those in φ.

It may be observed that this is the value of $\Pi(nc\varphi)$ of Le Gendre of the third species when $b = 1$.

—

SECOND SOLUTION.—*By Mr. B. McGowan, New-York.*

$\sin.^2\varphi = \frac{1}{2} - \frac{1}{2}\cos.2\varphi$, Hutton's Math. vol. ii. Formu-

lae xix, (Radius=unity) $\therefore$ by substitution $\frac{d\varphi}{1\pm n\sin.2\phi} =$ $\frac{d\varphi}{1\pm\frac{n}{2}\mp\frac{n}{2}\cos.2\varphi} = \frac{d\phi'}{a+b\cos.\phi'}$; by putting $2\phi=\phi'$, $2\pm n=$ a and $\mp n=b$; but by Hirsch's Integral Tables, page 274, $\int\frac{d\phi'}{a+b\cos.\phi'} = \frac{1}{\sqrt{(a^2-b^2)}}\times$arc. $\cos.\frac{b+a\cos.\varphi'}{a+b\cos.\varphi'}$; hence, by substitution, the integral of the proposed expression becomes known.

Mr. Capp's method of solution is similar to this; but his result is, the fluent of $\frac{d\varphi'}{a+b\cos.\varphi'} = \frac{1}{\sqrt{(a^2-b^2)}}$ arc. tan. $\frac{(a-b)\tan.\frac{1}{2}\varphi'}{\sqrt{(a^2-b^2)}}$; where a, b and φ' represent the same quantities, as above.

—

THIRD SOLUTION.—*By Mr. Charles Farquhar, Alex. D. C.*

Assume cotan. $\varphi=-x$, then $\frac{d\varphi}{\sin.^2\varphi} = dx$, and $d\varphi=dx$. $\sin^2\varphi=\frac{dx}{x^2+1}$, (because $\sin.^2\varphi=\frac{1}{\text{cosec.}^2\phi}=\frac{1}{x^2+1}$), $\therefore$ $\frac{d\varphi}{1\pm n\sin.^2\phi} = \frac{dx}{x^2+1\pm n}$. Hence, (putting $1\pm n=\pm a$) $\int\frac{dx}{x^2\pm a} = \frac{1}{\sqrt{a}}\times$arc, tan. $x\sqrt{\frac{1}{a}}$ rad. 1, or, $\frac{1}{2\sqrt{a}}\log.\frac{x-\sqrt{a}}{x+\sqrt{a}}$, according as the upper or lower sign is used. If now the values of a and x be replaced, the integral will be known.

—

FOURTH SOLUTION.—*By Mr. Charles Wilder, Baltimore.*

Put $t=\tan\phi$; then $d\varphi=\frac{dt}{1+t^2}$, and as square sec. is to square radius, so is square tan. to square sine; therefore, $\int\frac{d\phi}{1+n\sin.^2\varphi} = \int\frac{dt}{1+(1\pm n)t^2} =$ arc. (tan $= t$ and radius $\frac{1}{(1\pm n)^{\frac{1}{2}}}$).

Mr. Hallowell's solution is similar to this, but his result is $\int\frac{dx}{1\pm ax^2} =$

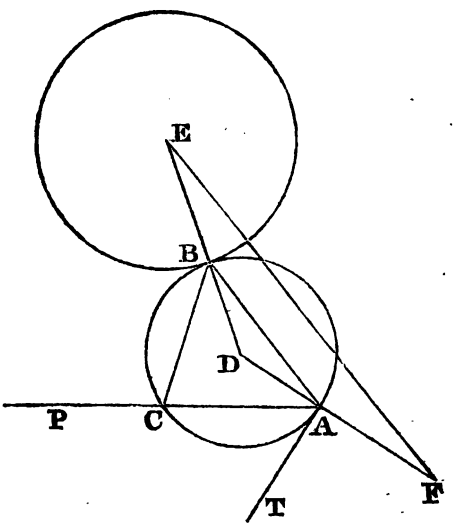
$\frac{1}{\sqrt{a}}$ arc (tan. $x\sqrt{a}$), or $\frac{1}{\sqrt{a}}$ log. $\frac{1+x\sqrt{a}}{\sqrt{(1-ax^2)}}$, according as a is affirmative or negative.

Question XV.—OR PRIZE QUESTION.

By Farrand N. Benedict, Montezuma, State of New-York.

The vertical angle of a triangle is given in magnitude, and the vertex of the triangle is in the circumference of a circle given in magnitude and position; also the base of the triangle is in a straight line given in position, and one extremity of the base is given in position. It is required to determine the triangle when its base is a minimum.

First Prize Solution,—*By Mr. Charles Farquhar, Alexandria, D. C.*



Construction. Let E be the centre of the given circle, AP the line given in position, and A the given point in it. Draw a line AT, making the angle PAT = the given vertical angle, and make AF perpendicular to AT, and = to

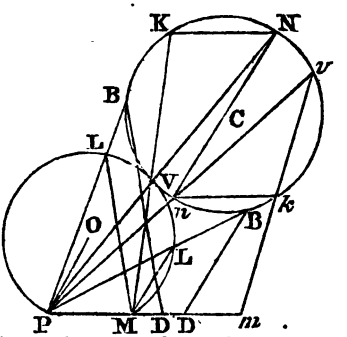
EB, the radius of the given circle ; join *FE*, and parallel to it, draw *AB*, cutting the given circle in *B*, through which, draw *ED*, to meet *FA* produced in *D*, in which, as a centre, with the distance *DA*, describe a circle, cutting *AP* in *C*; join *BC*, and *ABC* will be the triangle required.

Demonstr. Because *FE* is parallel to *AB*, and *FA* = *EB*, therefore, *AD* = *BD* ; consequently, the circle *ACB* touches the given circle at *B* ; and (3.32.) the angle *ABC* = *CAT* = the given angle. Now, as the chords of similar segments, are as their radii, and as *AD* is the radius of the least circle which can meet the given circle, therefore *AC* is a minimum.

Note. When the position of the line *AP*, and of the point *A* are differently assumed, the above construction would be easily varied to suit the different cases of the problem.

—

Second Prize Solution.—*By Mr. J. H. Swale, Liverpool, England.*



In the given circle, apply the chord *NK* (parallel to *PM*, the Line *given* in position) to subtend the given Angle, join *P* (the *given* point in *MP*) and *N* ; let *PN* cut the circle in *V*, and *KV* meet *PM* at *M* : the required triangle is *PVM*.

The angle *PVM* (= ∠*NVK*) being equal the given one, *V* in the circumference of the given circle, and *P* the given point ; it remains only to show, that *PM* is *less* than the Base (*PD*) of any triangle (*PBD*) whatever, having its vertex (*B*) in the circumference of the circle *NVK*, and the angle *PBD* equal *PVM* ; or the given angle.

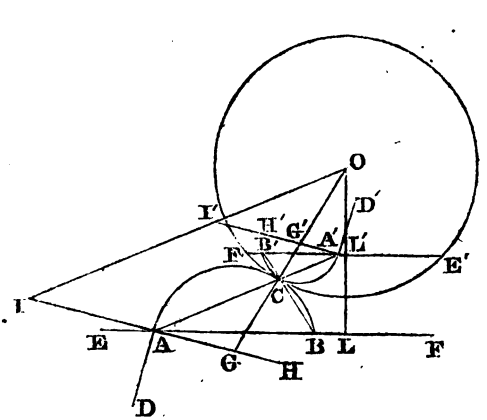
Through *P*, *M*, *V*, describe a Circle, intersecting *PB* at *L*, and draw *LM*. Because of the parallels *PM*, *NK*, the circles *touch* at *V*, or all the points *B* are *without* the circle *PVM*, or *PB* always *greater* than *PL* ; and (be-

cause of the equal angles *PVM*, *PBD*, *PLM*) the lines *BD*, *LM*, are par llel: that is, *PD* is always *greater* than *PM*; and consequently, *PM less* than *PD*: or *PM* a minimum.

The vertex *v* of the triangle *Pvm*, having the *maximum* base *Pm*, may be determined by applying *nK* in the other semicircle, and drawing *Pv* through *n*. Or, if *Nn* (a diameter of the given circle) be drawn, making, with *PM*, an angle equal Comp. of the given one; the position of the points *N*, *n*, determining the two limits, will be at once assigned.

When *V* is to be posited in the periphery of an Ellipse, or Curve of any order, given in position: we shall have to describe a circle through a given point (*P*), having its centre (*O*) in a right line (*PO*) given in position, and *touching* the Ellipse, or other Curve, given in position.

THIRD PRIZE SOLUTION.—*By Mr. Benjamin Hallowell, Alexandria, D. C.*



Let *CF'E'* be the given circle of which the centre is *O*, *EF** the line given in position, *A* the given point in *EF*, and *FAD* the given angle. Draw *AH* and *OL* at right angles to *AD* and *EF* respectively; then it is evident that the centre of the circle, circumscribing the required triangle, will lie in *AH*. On *HA* produced lay *AI* (in a direc-

tion from the perpendicular OL) equal to the radius of the given circle ; join IO, and draw AC parallel to it, also join OC, and produce it till it meet AH in G.; then, since $AI = OC$, therefore (2.6) $AG = GC$. Hence with the centre G and radius GA or GC, describe the segment ACB and join BC, and ABC will be the triangle required.

For (32-3) $< ACB = < BAD =$ the given vertical angle ; and since the radius of any segment whose centre is in AH, to pass through A and meet the given circle, must necessarily be greater than GA, it is evident that Ab is the least possible base.

Cor. 1. If A corresponded with L, AI might be laid either way, and there would be two similar and equal triangles formed.

Cor. 2. The trigonometrical calculation is extremely simple : for, if we join AO, we have in the triangle OAG, to determine AG ; whence the others are determined.

* If the corresponding prime letters are taken, the same construction and demonstration answer when the point is within the circle.

ACKNOWLEDGEMENT.

The following ingenious gentlemen, favoured the editor with solutions to the questions in Article IX. No. IV. The figures annexed to the names refer to the questions answered by each, as numbered in that article.ecl

Dr. Bowditch,Boston; Professor Strong, Hamilton College ; Charles Farquhar, Alexdria, D. C. ; Benjamin Hallowell, Alex. D. C. ; Charles Wilder, Baltimore, Maryland ; and Nemo, New-York; each most ingeniously answered all the questions.

Dubre Knight, Alex. D. C. answered all but 10, 14. John Capp, Harr. Penn. all but 10, 11, 12, 15 ; J. Ingersoll Bowditch, Boston, all but 5, 11, 13, 14, 15; Tyro, Lex. Kentucky, all but 10. 11, 13, 14, 15 ; Professor Bivver, S. C. College, Columbia, 1, 2, 3, 4, 6, 7, 8, 11 ; James Hamilton, Trenton, N. J. 1, 2, 3, 4, 5, 6, 7, 8 ; James Maccully, N. Brunswick, N. Jersey, 1, 2, 3, 4, 6, 7, 8, 10 ; James F. James, Trenton, Henry Darnall, Philadelphia, and Mathetus, Bucks Co. Penn. each answered, 1, 2, 3, 4, 6, 7, 8 ; Edward Giddins, Fort Niagara, and Gerardus B. Docharty, Flushing, L. I. each answered 1, 2, 3, 4, 6, 8; William F. Kells, Bergen. 1, 3, 4, 6, 7, 8 ; Mary Bond, Frederick, Maryland, 1, 3, 4, 6, 15 ; Devoor V Burger, N. Y. 1, 2, 3, 4, 5, 6 ; Daniel Shanley, Charleston, S. Carolina, James Sweeny, New-York, and William Vodges, Delaware County, Pennsylvania, each answered 1, 3, 4, 6, 8 ; James Farrell, New-York. 1, 3, 4, 10; John Swinburne, Brooklyn, L. I. 1, 3, 4; William A. W. Stigleman, 1, 4; Elias Lynch, New York, 9; H. Crosby Hyatts;town Maryland, 6; Farrell Ward, New-York; 1. 9 ; Michael Floy, New-York 4; Edward C. Ward, N Y. 10; Martin Roche, Phil. 11; Diarius, 13; Farrand N. Benedict, Montezuma, state of N. Y. Dubre Knight, Alex. D. C. Philomath, Albany, and D. Y. Y. Newport, Rhode Island, each most ingeniously answered the prize question.

Having found it impossible to give the preference to one of the three solutions to the Prize Question by Messrs. Charles Farquhar, Alex. D. Colum-

bia, J. H. Swale, Liverpool, England, and Benjamin Hallowell, Alex. D. C. each of them being excellent and fully deserving of the prize: the editor has published all as prize solutions, and the PRIZE has therefore been divided equally among those three ingenious and learned mathematicians.

The solutions to the Prize Question by Dr. Bowditch, Boston, Professor Strong, Hamilton College; Dubre Knight, Alexi D. C. Philomath, Albany; D., Y.Y. Newport, R. I. and Nemo, N. Y. were all ingenious and elegant.

ARTICLE XI.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN NO. VI.

QUESTION I. (69)—*By a correspondent.* N. Y.*

There are three horses, belonging to different men, employed to draw a load from Poughkeepsie to Hartford for 35 dollars. *A* and *B* are supposed to do $\frac{3}{11}$ of the work, *A* and *C* $\frac{5}{13}$, and *B* and *C* $\frac{2}{7}$ of it. They are paid proportionally; please divide their pay for them as it should be.

N. B. *He doubts the correctness of the answers to this question in Willet's Arithmetic. EDITOR.

QUESTION II. (70)—*By Messrs. Edward Evans and George Alsop, Philadelphia.*

Given the three following equations,

$$x^2+y^2+z^2=266\cdot$$
$$x^2+y+z^2=176\cdot5$$
$$(x+y+z)y=280;$$

to determine the values of x, y and z.

QUESTION III. (71)—*By Mr. Michael Floy, N. Y.*

Given $x+v=a$, $xyz+v=b$, $x+yzv=c$, and $xyzv+y=d$; to find the values of x, y, and v.

QUES. IV. (72)—*By Mr. J. Maccully, New-Brunswick, N. J.*

Given the area of a right-angled triangle, and the diagonal of its inscribed square to determine the triangle.

QUESTION V. (73)—*By Mr E. Giddins, Fort Niagara, N. Y.*

Required to find two cube numbers, such that the first multiplied into the product of their roots will be equal to the second, and the second multiplied into the product of their roots will be equal to 64 times the first.

QUESTION VI. (74)—*By the same.*

There are three numbers in geometrical progression, but if the third be diminished by the first, the result will be an arithmetical progression, and if the first and third be increased by the second, and the second by 2, the progression will be harmonical; required the numbers.

QUESTION VII. (75)—*By Nemo, New-York.*

Make x^2+xy+y^2 and x^2-xy+y^2 rational squares, if possible, if not, give a demonstration.

QUESTION VIII. (76)—*By J. Capp, Esq. Harr. Penn.*

To find four numbers, such that if the square of each be subtracted from their sum, the remainders shall all be squares.

QUESTION IX. (77)—*By Mr. Wm. Lenhart, York, Penn.*

If on the given base of (312) of a plane triangle, between two acute angles, a semicircle be described, the circumference of which cuts the other two sides; and there be given the straight line joining the points of intersection (120) and the perpendicular from the vertical angle on said line (85): it is required to determine the triangle.

QUESTION X. (78)—*By the same.*

In a combination lottery, according to the plan of Yates and McIntyre, consisting of 11 numbers or ballots, (and consequently 165 tickets,) if any combinations, viz: 2, 5, 9 and 4, 8, 11 be given, it is required to find their respective registers. Or, if the registers 86 and 123 be given to find their corresponding combinations.

QUESTION XI. (79).—*By Mr. Dennis W. Carmody, N. Y.*

To find an arc, such that if a perpendicular be dropt from the angular meeting of the cosine, and sine on the secant, the secant will be divided harmonically by said perpendicular and the arc.

QUESTION XII. (80)—*By Benjamin Hallowell, Alex. D. C.*

If from any point P the lines PA, PB and PC be drawn to the nearest corners of a square $ABCD$, the area of the triangle PAC formed by PA, PC, and the diagonal AC, will always be $\frac{1}{2}(PA^2+PC^2-2PB^2)$; required a geometrical demonstration; and from this property it is requir-

ed to determine the square by construction when these three distances are given.

QUESTION XIII. (81)—*By Mathetus, Bucks Co. Penn.*

In order to determine the altitude of an inaccessible object, I measured two lines, 3 and 4, forming a right angle, and from the three angular points, ascertained the angles subtended by the altitude of the object 30°, 35°, and 40° respectively : a solution is required.

QUESTION XIV. (82)—*By Mr. Elias Lynch, New-York.*

Integrate $c.\ \frac{dz}{dy} = \sqrt{(c^2+4y^2)}$.

QUESTION XV. (83)—*By the same.*

To find the equation and area of a curve whose subtangent $= \frac{2x(a-x)}{3a-2x}$.

QUESTION XVI. (84)—*By Mr. Charles Wilder, Baltimore.*

Integrate $\frac{2n\varphi y\,dy}{(1+ny^2)^{\frac{3}{2}}}$, y being the Sine of the arc. ϕ.

QUESTION XVII (86)—*By Professor Strong, Hamilton Col.*

Investigate the motion of the apsides in orbits differing infinitely little from circles, in a more concise manner than Newton has done in his ninth section, supposing that there is any number whatever of disturbing forces, and whether they are centrifugal, or centripetal ; the disturbing force being always supposed to act in the direction of the radius vector.

QUESTION XVIII. (87).—OR PRIZE QUESTION.

By Mr. J. H. Swale, Liverpool, England.

A right line is given in position : to assign the position of a point in the periphery of a circle (given in position and magnitude) such, that thence drawing given and equal tangents, and demitting perpendiculars (upon the line given in position) from the extremities of the tangents; the rectangle of those perpendiculars shall be given, or a maximum or minimum.

THE

MATHEMATICAL DIARY,

NO. VI.

BEING THE PRIZE NUMBER OF ROBERT ADRAIN, LL.D. Professor of Mathematics and Natural Philosophy, Rutger's College, New Brunswick, N. J.

ARTICLE XII.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XI. NO. V.

QUESTION I. (69.)—*By a Correspondent, New-York.*

There are three horses, belonging to different men, employed to draw a load from Poughkeepsie to Hartford for 35 Dollars. A and B are supposed to do $\frac{3}{11}$ of the work, A and C $\frac{5}{13}$, and B and C $\frac{2}{7}$ of it. They are paid proportionally: please divide their pay for them as it should be.

FIRST SOLUTION.—*By S———, New-York.*

Subtracting $\frac{3}{11}$, the joint shares of A and B, from $\frac{5}{13}$ those of A and C, we have the excess of C above B $=\frac{16}{143}$, and their sums by the question $=\frac{2}{7}$. Therefore the sum and difference of two quantities being given to find the quantities, we have the share of C $=\frac{109}{1001}$, and that of B $=\frac{87}{1001}$, which subtracting from $\frac{3}{11}$, gives that of A $=\frac{186}{1001}$ and $\frac{186}{1001}+\frac{87}{1001}+\frac{109}{1001}=\frac{472}{1001}$ their sum.

$$\frac{472}{1001} : 35 :: \begin{cases} \frac{186}{1001} : \$13.79 + \text{share of A,} \\ \frac{87}{1001} : 6.45 + \text{do B,} \\ \frac{109}{1001} : 14.75 + \text{do C.} \end{cases}$$

SECOND SOLUTION.—*By Mr. Ransford Wells, New-Brunswick, N. J.*

A+B proportional part of the work $=\frac{3}{11}=\frac{273}{1001}$,
A+C . . . . . $=\frac{5}{13}=\frac{385}{1001}$,
B+C . . . . . $=\frac{2}{7}=\frac{286}{1001}$;

$\therefore$ By addition, $2A+2B+2C=\frac{944}{1001}$,
and by division, $A+B+C=\frac{472}{1001}$;
But $A+B=\frac{273}{1001}$;

by subtraction, $C=\frac{199}{1001}$:
In like manner, we have $B=\frac{87}{1001}$,
and $A=\frac{186}{1001}$.
Hence, 472 : 199 : : 35 : $14\frac{247}{472}$,
472 : 87 : : 35 : $6\frac{218}{472}$,
472 : 186 : : 35 : $13\frac{374}{472}$,

Sum = $35.

The solutions of Messrs Gerardus B. Docharty, William Vogdes, Robert Parry, Selah Hammond, N. Leeds, and Joseph M'Keen, were similar to this.

[As the solutions of almost all the contributors bring out the same result as those given in Willet's Arithmetic, we must conclude that the answer to this question in that work is correct. Ed.]

QUESTION II. (70.)—*By Messrs. Edward Evans and George Alsop, Phil.*

Given the three following equations

$$x^2+y^2+z^2=266$$
$$x^2+y+z^2)y=176.5$$
$$(x+y+z)y=280;$$

to determine the values of x, y and z.

FIRST SOLUTION.—*By Mr. D. T. Disney, Cincinnati, Ohio,*

Subtracting the second equation from the first, we have $y^2-y=89.5$;

$\therefore$ by quadratics, $y=9.973+$.

Substituting this value for y in the third equation, and reducing, we shall have $x+z=18.112+$,

and $x=18.112-z$.

Now, by substituting this value of x, and the above value of y, in the second equation, we have

$328.044 - 36.224z + z^2 + 9.973 + z^2 = 176.5$;
or by reduction, $z^2 - 18.112z = -80.758$;
$\therefore$ by quadratics, $z = 10.171$ or 7.941.
Hence, the values of x, y, and z, are $x = 7.941$, $y = 9.973$, and $z = 10.171$.

This question was solved in a similar manner by Messrs. Enoch Laning, Robert Parry, James Sweeney, and Joseph McKeen.

SECOND SOLUTION.—*By Mr. E. Giddins, Fort Niagara.*

Subtracting the second equation from the first, we have $y^2 - y = 89.5$; whence $y = \frac{1}{2} \pm \sqrt{89.75} = a$; now, substituting this value for y in the first and third equations, transposing, &c., and we shall have

$$x^2 + z^2 = 266 - a^2,$$
$$x + z = \frac{280 - a^2}{a}.$$

Now, the sum and the sum of the squares of two numbers being given, the numbers may be easily found, by rules long since discovered.

This Question was solved in a similar manner by Mary Bond, Messrs. Daniel Shanley, James Hamilton, William Vogdes, Mathetus, and Devoor V. Burger.

THIRD SOLUTION.—*By William S. Smith, Natches, Mississippi.*

Take 2d equ. from 1st, we have $y^2 - y = 89.5$, solved by a quadratic, gives $y = 9.97365$; now from 2d equ. $x^2 + z^2 = 176.5 - y^2$ $\therefore x^2 + z^2 = 166.52635$; but from 3d equ. $xy + y^2 + yz = 280$, or $(x + z)\, y = 280 - y^2$ $\therefore (x + z) \cdot y = 180.7373$; put here the value of y, and dividing, we get $x + z = 18.1215$, squaring this, $x^2 + 2xz + z^2 = 328.37064$; subtract from this $x^2 + z^2 = 166.52635$ we have, $2xz = 161.84429$; take this from $x^2 + z^2$ and extract the square root, we get $x - z = 2.1638$. Hence by addition, subtraction, &c. we shall find

$$x = 10.1426, \text{ and } z = 7.9788.$$

This question was solved in a similar manner by Messrs. Ransford Wells, Selah Hammond, James Divver, Michael Floy, James O. Farrell, F. M. Noll, and William A. W. Stigleman. But it may not be improper to observe that these gentlemen, as well as several other contri-

butors, assumed for the 1st equation, $x^2+y^2+z^2=266.5$, which was in fact the equation given by the proposers in their original manuscript. The values of x, y and z, found, according to this supposition, are 10.5, 10 and 7.5.

FOURTH SOLUTION.—*By Mr. G. B. Docharty, Flushing.*

Given the three following equations $x^2+y^2+z^2+266=m$; $x^2+y+z^2=176.5=n$; and $(x+y+z)y=280=a$. Put $m-n=d$, then we have by subtracting the second equation from the first, $y^2-y=d$, or by quadratics $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{4d+1}$, the value of y, which put $=s$, hence by substitution

$s(x+z)=a-s^2$, or by division $x=z-\frac{a-s^2}{s}$, and $x^2=m-s^2-z^2$, or $x=\sqrt{(m-s^2-z^2}$:) put these two values of $x=$; then $\sqrt{(m-s^2-z^2)}=z-\frac{a-s^2}{s}$, or by squaring both sides $m-s^2-z^2=z^2-2z\frac{a-s^2}{s}+\frac{(a-s^2)^2}{s^2}$, and by transposition and division, $z^2-z\frac{a-s^2}{s}=m-s^2-\frac{(a-s^2)^2}{s^2}$. From which z may easily be obtained and then x.

QUESTION III. (71.)—*By Mr. M. Floy, New-York.*

Given $x+v=a$, $xyz+v=b$, $x+yzv=c$, and $xyzv+y=d$, to find the values of x, y, z and v.

FIRST SOLUTION.—*By Mr. J. Hamilton, Trenton, N. J.*

From the first equation $x=a-v$, by substituting this value for x in the third, $a-v+yzv=c$;

$\therefore yzv=c-a+v$, or $yz=\frac{c-a+v}{v}$. Substituting these values in the second equation, $(a-v)\cdot\left(\frac{c-a+v}{v}\right)+v=b$; hence we find $(2a-b-c)\ v=a^2-ac$, or $v=\frac{a^2-ac}{2a-b-c}$. But $x=a-v=\frac{a^2-ab}{2a-b-c}$. From the fourth equation by

substitution, $(a-v)(c-a+v)=d-y$; $\therefore y=d-(a-v)(c-a+v)=d-\left(\frac{a^2-ab}{2a-b-c}\right).\left(c-\frac{a^2-ab}{2a-b-c}\right)$.

SECOND SOLUTION.—*By Mr. James Sweeny, New-York.*

Subtracting the first equation from the second, we have, $xyz-x=b-a$,

$$\text{or, } yz=\frac{b-a+x}{x}$$

Again, from the first equation, $v=a-x$. Now, substituting these values of yz and v in the third equation, we have, $x+\left(\frac{b-a+x}{x}\right)(a-x)=c$

or $x^2+ab-a^2+2ax-bx-x^2=cx$

$$(2a-b-c)=x=a^2-ab,$$

$$\text{or } x=\left(\frac{a^2-ab}{2a-b-c}\right)$$

$$v=a-\frac{a^2-ab}{2a-b-c}=\frac{a^2-ac}{2a-b-c}$$

Now, by substituting this value of x in the above value of yz, we shall have the value of yz; then substituting those values of x, v, and yz in the fourth equation, we shall find the value of y, and hence, the value of z may be readily found.

THIRD SOLUTION.—*By Mr. W. Vogdes, Edgmont, Penn.*

From the first equation, $v=a-x$, from the second $v=b-xyz$, and from the third, $yz=\frac{c-x}{v}=\frac{c-x}{a-x}$; whence, $a-x=b-xyz$, or $xyz-x=b-a$. Substitute the value yz, above found, in this equation, and we have $\frac{cx-x^2}{a-x}-x=b-a$; from which $x=\frac{a^2-ab}{2a-b-c}$, $v=\frac{a^2-ac}{2a-b-c}$, and from these two values y may be readily found.

FOURTH SOLUTION.—*By N'Importe qui, New-York.*

Multiply the second equation by v, and the third by

x; and we have $xyzv+v^2=bv$, and $x^2+xyzv=cx$; subtract the latter equation from the former, and dividing the resulting equation by $x+v=a$, we have

$$v-x=\frac{bv-cx}{a};$$

thus, by adding and subtracting these two equations, and dividing the results by 2, we have $v=\frac{a^2-cx}{2a-b}=a-x$, and $x=\frac{bv-a^2}{2a+c}=a-v$; from which equations we find $x=\frac{a^2-ab}{2a-c-b}$, and $v=\frac{a^2-ac}{2a-b-c}$. Now, assume $x=g$ and $v=f$, then from the second and fourth equations, we find, $y=d-fb+f^2$

$$\text{and } z=\frac{b-f}{g(d-fb+f^2)}.$$

FIFTH SOLUTION.—*By S. of Brooklyn, Long-Island.*

The first equation taken from the sum of the second and third, gives $yzv+xyz=c+b-a$,

$$\text{or } (v+x)yz=c+b-a;$$

$$\therefore \text{ by substitution, } ayz=c+b-a,$$

$$\text{and } yz=\frac{c+b-a}{a}=d,$$

which value of yz substituted in the second equation, gives, $dx+v=b$,

$$\text{and } x+v=a;$$

$$\therefore \text{ by subtraction, } dx-x=b-a;$$

$$\text{or } x=\frac{b-a}{d-1}.$$

The rest may now be easily found.

QUESTION IV. (72.)—*By Mr. J. Maccully, N. Brunswick.*

Given the area of a right-angled triangle, and the diagonal of its inscribed square, to determine the triangle.

FIRST SOLUTION.—*By R. Parry, Vincent Town, N. J.*

Put the area of the triangle $=a$, the diagonal of its inscribed square $=b$, the base $=x$, and the perpendicular $=y$; and, because the diagonal $=b$, the side of the in-

scribed square $=\frac{b\sqrt{2}}{2}$, which we will call c. Then by similar triangles $x : y :: c : y-c$; and by mensuration $\frac{xy}{2}=a$, or $xy=2a$. Turn the preceding analogy into an equation, and it becomes $xy-cx=cy$, or, since $xy=2a$, $2a-cx=cy$, whence $y=\frac{2a-cx}{c}$. If we substitute this value of y in the equation $\frac{xy}{2}=a$, we obtain $\frac{2ax-cx^2}{2c}=a$, or, $2ax-cx^2=2ac$; hence $cx^2-2ax=-2ac$, and $x^2-\frac{2a}{c}x=-2a$, therefore $x=\frac{a}{c}\pm\sqrt{\left(\frac{a^2}{c^2}-2a\right)}$ The base and perpendicular being now known, the hypothenuse is easily found.

Second Solution.—*By William F. Kells, Bergen, N. J.*

Let $\frac{a}{2}$= the area of the triangle, b= the diagonal of the inscribed square, and put x= the perpendicular, then $\frac{a}{x}$= the base of the triangle. Now, it is well known, by similar triangles, that the side of the inscribed is equal to $\frac{ax}{x^2+a}$; $\therefore$ we have $\frac{ax}{x^2+a}=\sqrt{\left(\frac{b^2}{2}\right)}$ which equation, solved by quadratics, will give the perpendicular; hence, the triangle is easily determined.

Third Solution.—*By Mr. Devoor V. Burger, L. I.*

Let a=the area of the triangle, and d= the diagonal of the inscribed square; then $\frac{1}{2}d\sqrt{2}$= the side of the square. Let x= perpendicular of the triangle, then $\frac{2a}{x}$ = the base; therefore by similar triangles, $\frac{2a}{x} : x :: \frac{1}{2}d\sqrt{2} : x-\frac{1}{2}d\sqrt{2}$; which reduced gives the value of x= the perpendicular. Hence the base becomes also known.

FOURTH SOLUTION.—*By Juvenis, New-York.*

Let x= the base of the triangle, y= the perpendicular, m= the area; and since the diagonal of the inscribed square is given, the side can be easily found, which let $=a$. Now, by similar triangles, we have $x-a : a :: a : y-a$; $\therefore xy=a(x+y)$; but by hypothesis, $xy=2m$; hence $x+y=\frac{2m}{a}$. Now, having the sum and rectangle of the base and perpendicular given, each may be readily found by the resolution of these two equations.

The solutions of Messrs. Cyril Pascalis and Joseph McKeen, were similar to this.

QUESTION V. (73.)—*By Mr. E. Giddins, Fort Niagara, New-York.*

Required to find two cube numbers such that the first multiplied into the product of their roots will be equal to the second, and the second multiplied into the product of their roots will be equal to 64 times the first.

FIRST SOLUTION.—*By Mr. James O. Farrell, New-York.*

Let x^3 and y^3 be the given cubes, and x and y their roots, then $xy.(x^3)=y^3$, and $xy.y^3=64x^3$ per question. Whence dividing the first of these equations by y, we have $x^4=y^3$, and the second by x, we have $y^4=64x^3$, evolving this last and subtracting from the preceeding &c. we find $x^4-8x=0$.; hence by division and transposition $x^3=8$; and consequently the required numbers are 8 and 64.

SECOND SOLUTION.—*By Mr. N. Leeds, Philadelphia.*

Let x^3 and n^3x^3 be the cubes, then $x^2 \times x \times nx=n^3x^3$ per question; $\therefore x=n$; substitute in the next given equation, then $n^3x^3 \times x \times nx=64x^3$, or $x^6=64$ $\therefore x^3=8$, and $n^3x^3=64$, as required.

THIRD SOLUTION.—*By Mr. Alpheus Bixby, New-York.*

Put $x^3=$ the first and $y^3=$ the second cube number, then by the question $x^3xy=y^3$, and $y^3xy=64x^3$, by multiplication and transposition $x^4=\frac{y^3}{y}=y^2$ in the first,

and $y^4=\left(\frac{64x^3}{x}\right)64x^2$; by extracting the square roots $x^2=y$, and $y^2=64x$; $\sqrt{(y^2)}=\sqrt{(64x^2)}=8x$; but $y=x^2$; therefore $x^2=8x$ or $x=8$, and $(x^2)=64=y$.

FOURTH SOLUTION.—*By Mr. William A. W. Stigleman, Pennsylvania.*

If we assume, x^3 and y^3 for the numbers, we have, per question, $x^4y=y^3$ or $x^4=y^2$ or $x^2=y$. Again per question $xy^4=64x^2$, or $y^4=64x^2$; $\therefore y^2=8x$. Hence it is evident that if $x^2=y$, we shall have $x^4=8x$, or $x^3=8$; $\therefore x=2$ and $x^2=y$, hence $y=4$. Consequently the numbers are 8 and 64.

QUESTION VI. (74)—*Py Mr. W. E. Giddins, Fort Niagara, New-York.*

There are three numbers in geometrical progression, but if the third be diminished by the first, the result will be an arithmetical progression, and if the first and third be increased by the second, and the second by 2, the progression will be harmonical ; required the numbers.

FIRST SOLUTION.—*By Mr. Daniel Shanley, Charleston, S.C.*

Let x^2, xy, and y^2 represent the three numbers respectively, which fulfils the first condition ; then x^2, xy, and y^2-x^2 are in arithmetical progression ; $\therefore y^2=2xy$, or $y=2x$. Hence x^2, $2x^2$, and $4x^2$, will fulfil the first and second condition. Now, if x^2, $2x^2$, and $4x^2$ represent the three numbers respectively, it is only necessary to fulfil the third condition ; $\therefore 3x^2$, $2x^2+2$, and $6x^2$ are in harmonical progression ; hence $\frac{36x^4}{9x^2}=2x^2+2$, or $4x^2=2x^2+2$; $\therefore x=1$ and $y=2$. The numbers are 1, 2, and 4.

SECOND SOLUTION.—*By Mr. Enoch Laning, Bucks Co. Pen.*

Let x=first term and y=ratio them x, xy, xy^2 are the numbers, but per question x, xy, xy^2-x are in arithmetical progression ; $\therefore xy^2=2xy$, divide this by xy and we have $y=2$. Again, by question $3x$, $(2x+2)$, $6x$ are the numbers by harmonics $3x : 6x :: 3x-2x+2 : 2x+2-6x$, equating this proportion, transposing terms, dividing by

$6x$, we have $x=1$. Consequently 1, 2 and 4 are the numbers.

THIRD SOLUTION.—*By Mr. Solomon Wright, Bucks Co. Penn.*

Let x, xy and xy^2 represent the numbers in geometrical progression, then x, xy and $xy^2-x=$ arithmetical progression, and $x+xy$, $xy+2$ and $xy^2+xy=$ harmonical progression; consequently $xy-x=xy^2-x-xy$ or $y=2$, and $x+xy : xy^2+xy :: 2-x : xy^2-2$ and $2x^2y^2+x^2y-4xy+x^2y-2xy^2=2x$ by multiplying means and extremes and reducing $\therefore 18x^2=18x$ or $x=1$ by substituting for y its value and reducing; consequently the required numbers are 1, 2 and 4.

FOURTH SOLUTION.—*By Mr. Joseph McKeen, New-York.*

Let x, y, and z represent the numbers; then $xz=y^2$, $z=2y$ and $(x+y)\times(z-2)=(z+y)\times(2-x)$, per question. Now, because, $z=2y$, we have from the first equation, y $z=2x$; $\therefore z=4x$. Now, by substituting these values of y and z in the third equation, and reducing, we shall find $x=1$; $\therefore y=2$, and $z=4$.

FIFTH SOLUTION.—*By Mr. Cyril Pascalis, New-York.*

Let $x=$ the first and $y=$ the ratio; then x, xy, and xy^2, will represent the numbers in geometrical progression; then, x, xy, and xy^2-x, will be in arithmetical progression, $\therefore xy^2=2xy$, or $y^2-y=0$; $y=2$ or 0. Now, it is evident that x, $2x$, and $4x$ will satisfy the first and second conditions, $\therefore 3x$, $6x$, and $2x+2$ must be in harmonic progression, by the question; $\therefore 3x : 6x :: 2-x : 4x-2$, hence $x=1$, and the numbers required are 1, 2, and 4,

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SIXTH SOLUTION.—*By Mr. F. M. Noll, Harlaem, N. Y.*

Let x, y, and $z=$ the numbers, then $xz=y^2$, also by hypothesis $z=2y$; whence $y=2$, and by harmonical proportion,

$$3x ; 6x :: 2-x : 6x-2;$$
$$\therefore 12x^2-6x=12x-6x^2;$$

hence $x=1$, $y=2$, and $z=4$.

SEVENTH SOLUTION.—*By Mr. Michael Floy, New-York.*

Let $x+y$, x and $x-y$ denote any three numbers in arithmetical progression $\therefore$ then, by the question, $2x$, x, and $x-y$ are in geometrical progression; and $3x$, $x+2$ and $2x-y$ are in harmonical progression. Whence $2x(x-y)=x^2$ and $3x : 2x-y :: 2x-2 : y-x+2$. From the equation $2x^2-2xy=x^2$ or $x=2y$, substituting this value of x in the proportion, gives

$$6y : 3y :: 4y-2 : 2-y,$$
$$\text{or } 2 : 1 :: 4y-2 : 2-y;$$
$$\text{whence } 4-2y=4y-2, \text{ or } 6y=6,$$

and $y=1$. Hence $x=2y=2$, and the numbers are 4, 2, and 1.

QUESTION VII. (75.)—*By Nemo, New-York.*

Make x^2+xy+y^2 and x^2-xy+y^2 rational squares, if possible, if not, give a demonstration.

FIRST SOLUTION.—*By Professor Adrain.*

The first notice that I find of any question equivalent to the present is in the Arabian treatise of Algebra entitled K'hulasat-ul-Hisab, written by Baha-ul-din, who was born in 1575, and died in 1653, as appears from the extracts of the work made by Mr. Strachy.

At the conclusion of the Arabian work, the author mentions several problems of which the solution had been sought in vain by mathematicians, and among others gives the following:

To find three square numbers in geometrical proportion such that their sum may be a square. Now if we put z^2 and x^2z^2 for the first and second numbers, the third must be x^4z^2, and the sum $z^2+x^2z^2+x^4z^2$, and consequently x^4+x^2+1 must be a square, or if we put $\frac{x}{y}$ instead of x to embrace all numbers, we shall have $x^4+x^2y^2+y^4$ to be made a square, which is the formula to which the present question is reducible by multiplication.

Again, in Euler's Algebra vol. 2, page 112, English edition of 1810, that learned author mentions the formula x^4-x^2+1, and remarks that it can be proved that this formula can never be a square: and in a note on page 113,

by the very ingenious editor of that edition, it is shown that to make x^4-x^2+1 a square requires the formula p^2-q^2 and p^2+3q^2 be both squares, which is shown in a note at the end of the volume by the editor to be impossible. The method employed by this English editor to transform x^4-x^2+1 is exceedingly simple and perfectly applicable to x^4+x^2+1, which requires p^2+q^2 and p^2-3q^2 to be both squares, which is also shown in the last mentioned note to be impossible.

As this reduction of the proposed problem to another which has been already solved may not be entirely satisfactory to the readers of the Diary; I shall therefore give the problem an original solution without referring to any thing but known elementary principles.

If possible let the formulae x^2+xy+y^2 and x^2-xy+y^2 be both squares, and as every case may be reduced to that in which x and y are integers prime to each other, it will be sufficient to consider only numbers of that kind.

The formulae being by hypothesis both squares, their product must also be a square, therefore $x^4+x^2x^2+y^4=\square$: and therefore putting $\frac{x}{y}=z$, we have $z^4+z^2+1=\square$, the root of which must be expressed by z^2+x, in which x may be possitive or negative, therefore $z^4+z^2+1=z^4+2xz^2+x^2$ whence $z^2=\frac{1-x^2}{2x-1}$: and as x may be a fraction reduced to its least terms let $x=\frac{b}{a}$, and the equation becomes $\frac{x^2}{y^2}=\frac{a^2-b^2}{2ab-a^2}$, in which if $-b$ be put for $+b$ we have $\frac{x^2}{y^2}=\frac{b^2-a^2}{2ab+a^2}$, in which equations the fractions being in their least terms, the numerators must be equal, and also the denominators. Let us therefore consider the system of equations.

I. $x^2=a^2-b^2$. $y^2=2ab-a^2$.

To make a^2-b^2 a square it is well known that we must have :

(1) $a=m^2+n^2$, $b=m^2-n^2$, or (2) $a=m^2+n^2$, $b=2mn$, in which m and n must be prime to each other. Now m and n cannot be both odd, because a and b would have the

common divisor 2, therefore of m and n one must be even and the other odd; and as b in the second case must be even, therefore a is odd; and therefore in both cases a is odd; and consequently, in $y^2=2ab-a^2=a\times(2b-a)$ the factors are prime to each other; whence $a=\square$, $2b-a=\square$, and therefore in the

1st. case, (A) $m^2+n^2=\square$, $m^2-3n^2=\square$,

2d. case, (B) $m^2+n^2=\square$, $4mn-(m^2+n^2)=\square$.

The second of equations (B) being of the form $4N+3$, is evidently impossible: and in equation (A) we must have: 1. $m=p^2-q^2$, $n=2pq$;

or 2. $m=2pq$, $n=p^2-q^2$,

the numbers p and q being prime to each other.

In each case, $m^2+n^2=(p^2+q^2)^2$, and in

Case 1. $m^2-3n^2=(p^2+q^2)^2-16p^2q^2=\square$,

Case 2. $m^2-3n^2=(p^2+q^2)^2-(2p^2-2q^2)^2=\square$.

In Case 1 we must have $p^2+q^2=r^2+s^2$ and $4pq=2rs$, and in Case 2. $p^2+q^2=r^2+s^2$, and $2p^2-2q^2=2rs$. and thus in both cases we obtain $x=2rs$, $\sqrt{(r^4+r^2s^2+s^4)}$, and $y=r^4-s^4$. Now since x is evidently greater than either r or s, it is plain that if x and y were the least numbers making $x^4+x^2y^2+y^4$ a square, we should have from this value of x a new set of numbers fulfilling the same condition and less than x, which is impossible.

Next let us consider the formulae when x is negative.

II. $x^2=b^2-a^2$, $y^2=2ab+a^2$.

Now we must have 1. $b=m^2+n^2$, and $a=m^2-n^2$,

2. $6=m^2+n^2$, and $a=2mn$.

In each case, one of the two, m and n, must be even, and the other odd; therefore in the first case a and b are both odd, and in the second a is even and b is odd.

In case 1. $x^2=4m^2n^2$, $y^2=(m^2-n^2).(3m^2+n^2)$,

In case 2. $x^2=(m^2-n^2)^2$, $y^2=4mn.(m^2+mn+n^2)$, and thus we must have one or other of the systems.

(C) $m^2-n^2=\square$, $3m^2+n^2=\square$,

(D) $m^2-n^2=\square$, $4mn.(m^2+mn+n^2)^2$,

in each of which one of m and n must be even and the other odd.

In equations (C) assume 1. $m=p^2+q^2$, and $n=p^2-q^2$,

or 2. $m=p^2+q^2$, and $n=2pq$.

In the first case p and q cannot be both even or both odd, for then m and n would be both even; neither can

one of the two, p and q, be even and the other odd ; for then m and n would be both odd ; but m and n are one even and the other odd ; therefore the first assumption for making $m^2 - n^2$ a square, is inadmissible.

In the second case, m is odd and n is even ; and therefore $n^2 + 3m^2$ is of the form $4N+3$, and therefore cannot be a square.

Lastly, let us examine the system D.

The second square gives $mn \times (m^2 + mn + n^2) = \square$, and as m and n has no common divisor, each of them must be a square ; and therefore putting $m = p^2$, $n = q^2$, we have $m^2 + mn + n^2 = p^4 + p^2q^2 + q^4 = \square$.

This last formula being the same as $x^4 + x^2y^2 + y^4$, we prove as before that the equation is impossible.

Thus it is completely demonstrated that the formulæ $x^4 + xy + y^2$, and $x^2 - xy + y^2$, cannot be both squares with the same values of x and y.

This demonstration differs essentially from that derived from the method of the editor of Euler's Algebra, edition of 1810. According to that method, the possibility of the equations employed $p^2 + q^2 = \square$, $p^2 - 3q^2 = \square$, would by no means be sufficient to demonstrate the possibility of the equation $x^4 + x^2y^2 + y^4 = \square$; but in the present demonstration, the possibility of the equations employed would prove the possibility of the proposed formulæ.

SECOND SOLUTION.—*By Dr. Henry J. Anderson, Professor of Mathematics, Columbia College, New-York.*

It is not possible for the following reasons. A slight examination is sufficient to show that if ever possible, it must be so also in positive and incommensurable integers. Now x and y cannot both be odd ; for as x^2, y^2, and $x^2 + xy + y^2$ must be of the form $8n+1$, xy must be of the form $8n+7$, and therefore the second formula, of the form $8n+3$, which can never be a square. Let x then be even and y odd. As the product of the two formulæ, that is, $x^4 + x^2y^2 + y^4$, or $(x^2 + y^2)^2 - x^2y^2$ must be an odd square, we must suppose $x^2 + y^2 = p^2 + q^2$, and $xy = 2pq$. Now, by a mode of reasoning in all respects similar to that employed by Euler, Vol. 2. sect. 230, it is proved that this requires us to have $r^2 - s^2$, and $r^2 - 4s^2$ separately odd

squares, where s is even, and r and s divisors of x and y. In the same way it may be shown that in order to make these last two formulæ separately odd squares, we must make t^2-u^2 and t^2-4u^2 separately odd squares, where u is even and t and u divisors of r and s. In this way, we must come at last to formulæ v^2-w^2, v^2-4w^2 to be odd squares, where $\frac{1}{2}w$ could no longer have divisors, in which case v^2-w^2 could not possibly be an odd square. Therefore x^2+xy+y^2 can never be squares at the same time.

THIRD SOLUTION.—*By Professor Strong, Hamilton College, New-York.*

I suppose that this question requires that the two expressions should both be squares with the same values of x and y; $\therefore$ their product $(x^2+y^2)^2-x^2y^2$ must be a square also, put it equal to a^2; then $(x^2+y^2)^2-x^2y^2=a^2$; $\therefore (x^2+y^2)^2=a^2+x^2y^2$. Hence $a^2+x^2y^2$ must be a square, and $(x^2+y^2)^2-4x^2y^2=(x^2-y^2)^2 \Rightarrow a^2-3x^2y^2$; $\therefore$ $a^2-3x^2y^2$ must be a square also; $\therefore$ if the two formulæ given in the question can both be squares with the same values of x and y, then are the two formulæ $a^2+x^2y^2$ and $a^2-3x^2y^2$ both capable of being squares with the same values of a and xy; but the two formulæ $a^2+x^2y^2$ and $a^2-3x^2y^2$ cannot both be squares with the same values of a and xy, as is shown in Euler's Algebra; (see Vol. II of that work, Appendix, page 481, second English edition;) hence as the two expressions $a^2+x^2y^2$ and $a^2-3x^2y^2$ cannot both be squares with the same values of a and xy; therefore the two formulæ given in the question cannot both be squares with the same values of x and y.

The solutions of Messrs. James Divver, S. C. Col. Columbia, and James Maccully, N. Y. were similar to this.

QUESTION VIII. (76.)—*By John Capp, Esq. Harrisburg, Penn.*

To find four numbers, such that if the square of each be subtracted from their sum, the remainders shall all be squares.

FIRST SOLUTION.—*By Mr Charles Farquhar, Alex. D C.*

Let w, wx, wy, and wz, be the numbers, and put, $1+$

$x+y+z=s$; then $ws-w^2=\square=m^2w^2$, or $s=(m^2+1)w$, and we have to make m^2+1-x^2. m^2+1-y^2, and m^2+1-z^2, all squares; to do which, assume in the first formula, $x=m$, in the second, $y^2=2m$, and to make the 3d a square, assume $m^2+1-z^2=\{1+p(m-z)\}^2$; then $z=\frac{(p^2-1)m+2p}{p^2+1}$, p and m may be taken at will, so that m is $\frac{1}{2}$ any square. If $m=8$, and $p=3$; then $x=8$, $y=4$, $z=7$, and $s=20$; also, $w=\frac{s}{m^2+1}=\frac{4}{13}$, and the required numbers are, $\frac{4}{13}$, $\frac{18}{13}$, $\frac{28}{13}$, and $\frac{32}{13}$.

Second Solution.—*By Mr. William Lenhart, York, Penn.*

Take any two squares, whose roots are m^2 and $2m^2$, and divide their sum into two other squares whose roots are $2m^2-n^2$ and $2nm$. Let now $(4m^4+n^4)x^2$ represent the sum of the four numbers required: and let the first number be denoted by n^2x. the second by $2n^2x$, the third by $(2m^2-n^2)x$, and the fourth by $2nmx$, and all the conditions of the question will be fulfilled. And since $(4m^4+n^4)x^2=$ sum of the four numbers, we shall have $(n^2+2m^2+2m^2-n^2+2mn)x=(4m^4+n^4)x^2$, or $x=2m.\frac{2m+n}{4m^4+n^4}$; where any numbers whatever, provided m be greater than n, may be substituted for n and m. If $n=1$ and $m=2$, then $x=\frac{4}{113}$, and the numbers are $\frac{4}{113}$, $\frac{16}{113}$, $\frac{28}{113}$, and $\frac{32}{113}$.

Third Solution.—*By Mary Bond, Frederick'stown. Md.*

Let x, y z and v, be the numbers; then $x+y+z+v-x^2$; $x+y+z+v-y^2$; $x+z+y+v$ z^2, and $x+y+z+v-v^2$ are to be squares. Suppose $x+y+z+v=a^2$, and $a^2-x^2=\overline{a-nx}^2$ we shall then find $x=2a\times\frac{n}{n^2+1}$. In the same manner we may find $y=2a\times\frac{m}{m^2+1}$, $z=2a\times\frac{p}{p^2+1}$ and $v=2a\times\frac{r}{r^2+1}$. Now $x+y+z+v=2a\times\frac{n}{n^2+1}+\frac{m}{m^2+b}+\frac{p}{p^2+1}+\frac{r}{r^2+1}=a^2$; whence $a=2\times\frac{n}{n^2+1}+$

$\frac{m}{m^2+1}+\frac{p}{h^2+1}+\frac{r}{r^2+1}$, where any numbers may be substituted for n, m, p & r, and thence x, y, z & v, become known. From considering the above solution we may solve the question thus: The number 5 is equal to 1^2+2^2, and also equal to $\left(\frac{2}{5}\right)^2+\left(\frac{1}{5}\right)^2$. Hence if $5a^2$ represent the sum of the four numbers, and a the first number, $2a$ the 2d, $\frac{2a}{5}$ the 3d, and $\frac{11a}{5}$ the 4th, all the conditions will be fulfilled; and as $a+2a+\frac{2a}{5}+\frac{11a}{5}=5a^2$, we have $a=\frac{5}{12}$ and consequently, $\frac{5}{12}$, $\frac{10}{12}$, $\frac{1}{12}$, and $\frac{30}{12}$, for the four numbers required.

FOURTH SOLUTION.—*By Mr. Farrand N. Benedict, Montezuma, New-York.*

Put s^2 = the sum of the required numbers x, y, z, and v. Divide according to the common rules (s^2) into two square numbers, whose roots may be of the form $\frac{2ns}{n^2+1}, \frac{n^2s-s}{n^2+1}$; and also into two other square numbers whose roots may be of the form $\frac{2ts}{t^2+1}, \frac{t^2s-s}{t^2+1}$. It is obvious that it only remains to make the sum of those roots equal to s^2. Equating and reducing, we have $\frac{2n+n^2-1}{n^2+1}+\frac{2t+t^2-1}{t^2+1}=s$; where n and t may be assumed any numbers greater than unity. If we assume $n=2$ and $t=5$, we find $x=\frac{704}{845}$, $y=\frac{528}{845}$, $z=\frac{288}{845}$, and $v=\frac{352}{845}$. In the same manner, we might find more numbers answering the conditions of the problem.

FIFTH SOLUTION.—*By Dr. Henry J. Anderson, N. Y.*

The following method will enable us to find not only four, but any number of numbers, answering the conditions of the question.

Let s = sum of the numbers, then since s must be equal to the sum of two squares, as many ways as there are numbers, put $s=x^2+y^2$; then (Eul. Vol. II. sect. 219) s may be divided into any number of pairs of other squares $x'^2+y'^2$, $x''^2+y''^2$, &c. by putting

$$x'=\frac{2mn}{m^2+n^2}y-\frac{m^2-n^2}{m^2+n^2}x$$

$$y'=\frac{2mn}{m^2+n^2}x+\frac{m^2-n^2}{m^2+n^2}y$$

$$x''=\frac{2pq}{p^2+q^2}y-\frac{p^2-q^2}{p^2+q^2}x$$

$$y''=\frac{2pq}{p^2+q^2}x+\frac{p^2-q^2}{p^2+q^2}y, \text{ \&c. \&c.}$$

where m, n, p, q, &c. are arbitrary.

Moreover, $x+x'+x''+$ &c.$=s=x^2+y^2$, Adding together the values of x, x', x'', &c. and calling the coefficients of x and y, a and b, we have

$$x^2+y^2=ax+by;$$

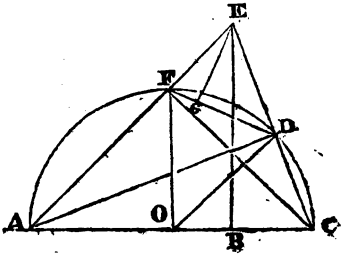
where, in order that x shall be rational, $4by-4y^2+a^2$ must be a square. This, by the common method, is effected by putting $y=\frac{ac+b}{c^2+1}$, which gives us $x=cy$, or $a-cy$, where c may be taken at pleasure.

To solve the case of four numbers, let $m=2$, $n=1$, $p=1$, $q=2$, $x=1$, $s=3$; then $a=\frac{9}{5}$, $b=\frac{11}{5}$; and putting $c=\frac{1}{3}$ (to obtain positive and different nnmbers), we shall find $x=\frac{24}{25}$, $y=\frac{63}{25}$, whence $x'=\frac{36}{25}$, $x''=\frac{324}{125}$, $x'''=\frac{57}{25}$, $s=\frac{999}{125}$. If $c=1$, the four numbers will be 2, $\frac{2}{5}$, $\frac{14}{5}$, and $\frac{14}{5}$, and $s=8$.

QUESTION IX. (77.)—*By Mr. Wm. Lenhart, York, Penn.*

If on the given base (312) of a plane triangle, between two acute angles, a semicircle be described, the circumference of which cuts the other two sides; and there be given the straight line joining the points of intersection (120) and the perpendicular from the vertical angle on said line (85): it is required to determine the triangle.

FIRST SOLUTION.—*By Mr. James Macculley, New-York.*



Let AEC be the triangle having the base AC=312, FD= 120, and EG = 85. Let fall the perpendicular EB, and join FC and DA. The triangles AEC and EFD are similar; hence FD : EG :: AC : EB =221, and the area of the

triangle AEC $=\frac{AC \times EB}{2}$ =34476, also CA : AE :: FD : DE ; or 312 : 120 :: AE : DE $= \frac{5}{13} \times AE$. Put therefore DE $=5x$ and AE $= 13x$; then AD $= 12x$, for the same reason if we put FE$=5y$, then CE $= 13y$; hence $\frac{AD \times CE}{2}$ =34476 or $78xy=34476$, or $xy = 442$, and (Euc. 13. 2) $CE^2+2CA \times AE=AC^2+AE^2$ or $169y^2+338x^2-130xy=312^2+169x^2$ or $y^2+2x^2-\frac{1}{13}xy=576+x^2$ or $y^2+x^2-\frac{1}{13}yx=576$. But $x=\frac{442}{y}$ and by substitution $y^2+\frac{195364}{y^2}-340=576$ or $y^4-916y^2=-195364$; hence $y=18.38722$, and $x=24.04163$, and AE$=13x=312.541$ and CE$=13y=239.03386$ as required.

SECOND SOLUTION.—*By Mr. Selah Hammond, Brooklyn, L. I.*

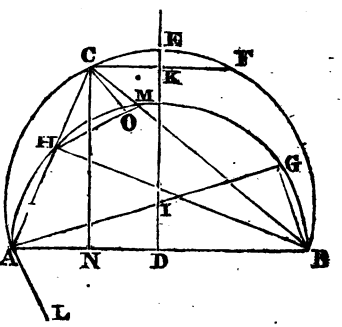
By a familiar process it may be demonstrated (see the above diagram,) that EFD is equal to the angle ACE, and the angle FDE is equal to the angle EAC ; hence it is evident that the two triangles ACE and FDE are similar, the angle at E being common to both. Therefore, as 120 : 312 or 5 : 13 : EF : EC, and 5 : 13 : : ED : EA. Therefore, put EF$=5x$, and EC$=5y$; then AE$=13y$ and CE$=13x$; join AD and CF. Then, as the angles ADC and AFC are in a semicircle, they are therefore right angles. Consequently EFC and EDA are also right angles ; $\therefore 169x^2-25x^2=144x^2=FC^2$, and FC$=12x$; in the same manner AD$=12y$. The perpendicular height of the whole triangle is found thus ; as 120 : 312 : : 85 : 221 ; whence the area$=\frac{312 \times 221}{2}=34476$. Therefore $\frac{AE \times FC}{2}=\frac{12x \times 13y}{2}=78xy=34476$, and $xy=442$. Then as the rectangles of a quadrangle inscribed in a circle is equal to the sum of the rectangles of its opposite sides, we have $12x \times 12y=144xy=(13x-5y) \times (13y-5x)+120 \times 312=194xy-65x^2-65y^2+37440$. By transposing $65x^2+65y^2=50xy+37440$, and putting 22100 for its equal $50xy$, and dividing the whole by 65, we obtain $x^2+y^2=916$,

from which and the former equation $xy=442$, the values of x and y are readily obtained. x being found $=17\sqrt{2}$, and $y=13\sqrt{2}$; whence BC$=169\sqrt{2}$, and AC$=221\sqrt{2}$.

THIRD SOLUTION.—*By Mr. James Dwyer, S. C. College, Columbia.*

If AAEC (see the figure to the first solution,) be the triangle on the given base, (312) AC, DF$=120$, and EC$=85$. The triangles AEC and EFD are equiangular, having the angles A and C respectively equal to the angles FDE and DEF; $\therefore$ DF : EC :: AC : EB; that is, 120 : 85 :: 312 : 221$=$EB, the perpendicular from the vertical angle on the base AC. From the centre O draw the radii OD and OF, and join CF. All the sides of the triangle FOD are given. Consequently all the angles are given. The angle FCE$=$half the angle FOD, and CFE being a right angle, the vertical angle FED or AEC is given$=67°23'$ nearly. Now, having the base, perpendicular, and vertical angle, the triangle can be constructed by Prob. V. Appendix to Simpson's Algebra.

FOURTH SOLUTION.—*By Thomas J. Matthews, Professor of Mathematics, Trans. Col. Lexington, Kentucky.*



On the given base AB, as a diameter, describe the semicircle AH MGB, with the given distance between the points of intersection, from B cut off the arc BG, join AG, from the centre D, of the circle AMB, draw DE perpendicular to the diameter AB, from I, the intersection of DE with AG, as a centre, with radius AI, describe an arc AEB, on DE; take DK a fourth proportional to the distance between the points of intersection, the given perpendicular thereon, and the base, draw FKC parallel to AB, from C,

or F, its intersections with the arc AEB, draw lines to A and B, and ABC will be the required triangle.

DEMONSTRATION. Conceive the triangle constructed join BG, BH, and draw AL perpendicular to AG, then AL is a tangent to the circle AEB, at the point A, consequently the angle BAL=angle ACB, but BAL=ABG; also BHC being a right angle, CBH; is the complement of ACB, and for the same reason BAG is the complement of ABG=ACB; therefore CBH=BAG, and being both angles at the circumference, they intercept equal arcs, that is, the arcs BG and HM are equal, or HM=the given distance between the points of intersection of the circle with the sides of the triangle. Join HM, draw CN parallel to DE, and co perpendicular to HM; then ABC and CHM being similar triangles, their sides and altitudes are proportional, that is, HM : CO : : AB : CN =DK; co is therefore equal to the given perpendicular on the distance between the points of intersection.

QUESTION X. (78.)—*By Mr. William Lenhart, York, Penn.*

In a combination lottery, according to the plan of Yates and McIntyre, consisting of 11 numbers or ballots, (and consequently 165 tickets.) if any combinations, viz; 2, 5, 9 and 4, 8 11 be given, it is required to find their respective registers. Or, if the registers 86 and 123 be given to find their corresponding combinations.

FIRST SOLUTION.—*By Professor Adrain, Rutger's College, N. B N. Jersey.*

According to the method of Messrs. *Yates* and *Mc Intyre*, a series of the natural numbers is assumed from 1 to any given number n, which, in the present example, is 11; any three of those numbers constitute a ticket; and being arranged according to their quantity, are regularly numbered from the least, which is 1, 2, 3, to the greatest 9, 10, 11, and the number thus belonging to each ticket is called its register.

Now, let a, b, c, be the three numbers of a ticket, R= the register corresponding, and by an easy calculation of the combinations of the numbers; we have $R=\frac{1}{6}.n.(n-1).(n-2)-\frac{1}{6}(n-a-1)(n-a).(n-a+1)+\frac{1}{2}(2n-a-b).(b-a-1)+c-b$. From this general formula the

register R may be found when n, a, b, c are given, and conversely, if n and R be given, the values of a, b, c, may be found.

Exam. 1. Let $n=11$, $a=2$, $b=5$, $c=9$, then $R=\frac{1}{6}.11.10.9-\frac{1}{6}.8.9.10+\frac{1}{2}(22-7).2+9-5=165-120+15+4=64$.

Exam. 2. Let $n=11$, $R=86$, also put $b=a+h$, $c=b+k$; then by substitution in the general equation we have, $86=165-\frac{1}{6}(10-a).(11-a).(12-a)+\frac{1}{2}(22-2a-h).(h-1)+k$; whence $\frac{1}{6}.(10-a).(11-a).(12-a)=79+\frac{1}{2}(22-2a-h).(h-1)+k$. Now the least values of h and k being each unity, we have in this case

$$\frac{1}{6}(10-a).(11-a).(12-a)=80$$
$$\text{or } (10-a).(11-a).(12-a)=480;$$

and the value if a that satisfies this equation or first gives a greater number than 480, will be the required value of a.

If $a=4$, we have $(10-a).(11-a).(12-a)=6.7.8.=336$, therefore a is less than 4.

If $a=3$. $(10-a).(11-a).(12-a)=7.8.9=504$. therefore $a=3$, which being substituted in the last equation containing a, h, k, A becomes

$$k=5-\tfrac{1}{2}(16-h)(h-1).$$

Since $a=3$, and the greatest value of h is so, therefore h is not greater then 7, and k being positive, we must have $h=1$; whence $k=5$; therefore $D=3+1=4$, and $C=4+5=9$.

Then the required combination is 3, 4, 9, answering to the register 86.

SECOND SOLUTION.—*By Professor Matthews, Lex. Ken.*

Let n=number of ballots, and m, p, q, the three figures of any combination in the same order in which they occur, that is, m the first, p the second, q the third; now it is evident that m changes its value after $m-1$ terms of the series $\frac{n-1.n-2}{2}+\frac{n-2.n-3}{2}+\frac{n-3.n-4}{2}$ &c. that p changes its value after $p-m-1$ terms of the series $(n-m-1)+(n-m-2)+(n-m-3)$ &c. and that q increases with the increase of $q-p$; but the sum of the first series is $\frac{(n-m+1)(m-1)(m-2)}{2}+\frac{(m-1)(m-2)(m-3)}{6}$

and that of the second is $\frac{(2n-m-p)\,(p-m-1)}{2}$, consequently the register number is $=\frac{(n-m+1)\,(m-1)\,(n-2)}{2}+\frac{(m-1)\,(m-2)\,(m-3)}{6}+\frac{(2n-m-p)\,(p-m-1)}{2}+q-p$; substitute the numbers 2, 5, 9, and 4, 8, 11, in this formula, and we obtain 64 and 127 for the respective registers.

To determine the combination when the register is given, sum the terms of the series $\frac{(n-1)\,(n-2)}{2}+\frac{(n-2)\,(n-3)}{2}+$ &c. until another term added would render the amount equal to or greater than the register number; the number of terms thus sumed being $m-1$, m is determined; then sum the terms of the series $(n-m-1)+(n-m-2)+$ &c. until another term added, would with the former series make an amount equal to, or greater than the register, the number of terms of the latter series being $p-m-1$, p is determined; then the sum of the two series substracted from the register number leaves a remainder $=q-p$, which determines q; thus for the register 86, we have $45+36=81=$sum of the first series; therefore $m-1=2$, $m=3$, again $n-m-1=11-3-1=7$, and $81+7=88>86$, consequently $p-m-1=o$, $p=m+1=4$ and $86-81=5=q-p$, whence $q=5+4=9$; 3, 4, 9 is therefore the combination. For 123, we have $45+36+28=109$, and $m=4$, then $109+6+5=120$, and $p=7$, again $123-120=3$ and $q=10$, consequently 4, 7, 10 is the combination answering to the register 123.

THIRD SOLUTION.—*By Mr. Charles Farquhar, Alex. D. C.*

By an inspection of the law of combinations, we may readily obtain the following general formula. Put R= the register, and m, n, and p, the corresponding combination; then $R=165-\frac{(9-m)\,.\,(10-m)\,.\,(11-m)}{1\,.\,2\,.\,3}-\frac{(11-n)\,.\,(12-n)}{1\,.\,2}+p-n.$

If $m=2$, $n=5$, and $p=9$; then, R=64. If $m=4$,

$n=8$, and $p=1$; then R=127. If the register be given, and the combination be required, the above equation must be resolved by indeterminate analysis. For example, if R=86; then is found, $m=3$, $n=4$, and $p=q$. If R=123; then, $m=4$, $n=7$, and $p=10$.

FOURTH SOLUTION.—*By Dr. Henry J. Anderson.*

Let b =n° of ballots,

t =n° of tickets,

c, c', c'', the combination numbers to find the register r.

Then, by applying the known formulæ for combinations,

$$t=\tfrac{1}{6}b(b+1)(b+2)$$

$$r=t-\tfrac{1}{6}(b-c)(b-c-1)(b-c-2)$$
$$-\tfrac{1}{2}(b-c')(b-c'+1)+c''-c'$$

The registers for 2, 5, 9, and 4, 8, 11, will thus be found to be 64 and 127.

To find the combination numbers corresponding to the registers 86 & 123, compare them with the terms of the series

1, 46, 82, 110, 131, 146, 156, 162, 165,

terms which the preceding formula shows to belong to the combinations 1, 2, 3; 2, 3, 4; 3, 4, 5, &c. whence it will easily appear that the required combination numbers are, 2, 3, 8, and, 4, 7, 10.

QUESTION XI. (79.)—*By Mr. Dennis W. Carmody, N. Y.*

To find an arc such that if a perpendicular be dropt from the angular meeting of the cosine and sine on the secant, the secant will be divided harmonically by said perpendicular and the arc.

FIRST SOLUTION.—*By Professor Adrain.*

Let radius =1, x= cosine of the arc required; then x^2= the segment between the centre and perpendicular, and $1-x^2$=middle segment; also $\frac{1}{x}$=secant, and $\frac{1-x}{x}$= the greatest extreme: then by harmonic proportion,

$$\tfrac{1}{x} : \frac{1-x}{x} :: x^2 : 1-x^2,$$

Whence $x^2(1-x)=1-x^2$;

which cubic gives for our root $x=1$. And dividing by $1-x$, we have the quadratic,

$$x^2 = 1 + x,$$

whence $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$;

of which two values there is only one applicable to the circle, viz. $x = \frac{1}{2} - \frac{1}{2}\sqrt{5} = -.6180340$, which is the cosine of an indefinite series of arcs of which the least is 128° 10′ 22″.

The root $x = 1$, gives the arc $= 0$; there is therefore no finite arc in the first quadrant that satisfies the question.

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SECOND SOLUTION.—*By Professor Strong, Hamilton Col.*

Let φ denote the arc to radius unity; then (per quest.) we have

$$\frac{1}{\cos\phi} : \frac{1}{\cos\varphi} - 1 :: \cos^2\phi : 1 - \cos\varphi,$$

or, $1 : 1 :: \cos^2\varphi : 1 + \cos^2\varphi$;

hence $\cos^2\phi - \cos\varphi = 1$; $\therefore \cos\varphi = 0.5 - \sqrt{1.25} = -0.61803$, and $\phi = 128° 10′ 20″$ very nearly.

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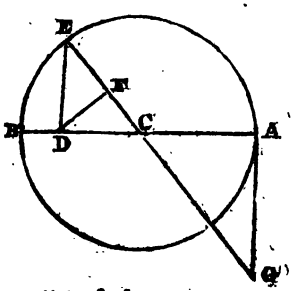
THIRD SOLUTION.—*By Dr. Henry J. Anderson.*

If by this be meant that the *segments* in their natural order, are to be in harmonical proportion, we shall then have an equation of the fourth order $x^4 + x^3 - 3x^2 + 1 = 0$, to determine x the cosine. If all the segments are to be reckoned from one and the same extremity of the secant, then no arc can be found answering the condition of the question; for the quadratic equation which results, gives two values for the cosine, one of which is greater than radius, and the other is negative. Now, if the problem be possible at all, it must be possible in the first or fourth quadrant, because in the second and third, the perpendicular would fall on the secant produced. The negative root would have solved the question, if it had been proposed in this form; "to find an arc such that if a perpendicular be dropped from the angular meeting



of the cosine and sine on the secant or secant produced, the secant, the segment between the perpendicular and tangent, and the segment between the variable extremity of the arc and the tangent shall be in harmonical proportion." With CB any radius describe the circle BEA. Let A be the origin of the arcs. Divide BC in extreme and mean ratio in D. Draw DE at right angles to BC, AE will be the arc required. Draw EG through C, and AG and DF at right angles to AC and CF. Because DC is a mean proportional between DB, BC, and also between FC, CE; BD=CF and DC=EF. Now GE : GC : : AD : AC : : CB : CD : : CB : DB : : EF : FC. Therefore GE : GC : : EF : CF, which is what is now required by the problem.

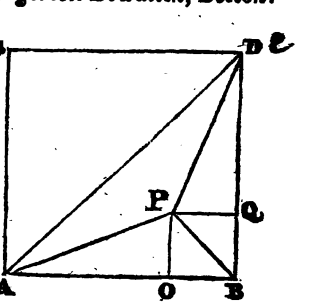
Cor. DC=2. cos. 72°. therefore AE=128° 10′ 22″.

QUESTION XII. (80.)—*By Mr. Benjamin Hallowell.*

If from any point P the lines PA, PB, PC, be drawn to the nearest corners of a square ABCD, the area of the triangle PAC formed by PA, PC, and the diagonal AC, will always be $\frac{1}{4}(PA^2+PC^2-2PB^2)$; required a geometrical demonstration; and from this property it is required to determine the square by construction when these three distances are given.

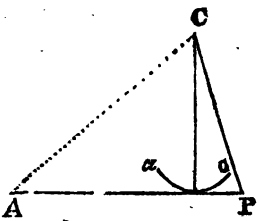
FIRST SOLUTION.—*By Mr. J. Ingersoll Bowditch, Boston.*

Construct the square ABCD, and from any point P draw PA, PB and PC; also the diagonal AC. From P drop the perpendiculars PO and PQ. And, as the area of ABC=$\frac{AB\times BC}{2}$. and APB=$\frac{AB\times OP}{2}$, and of BPC=$\frac{BC\times PQ}{2}$.



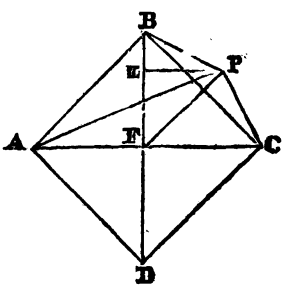
Hence the area of APC=$AB\times\frac{(AB-PQ-OP)}{2}=\frac{AB^2-AB\times PQ-AB\times OP}{2}$, and since $AP^2=PB^2+AB^2-2AB\times OB$; $\therefore AB\times OB=\frac{PB^2+AB^2-AP^2}{2}$; substitu-

ting this in the above equation and reducing, gives $\frac{1}{4}(AP^2 - PB^2 + AB^2 - 2AB \times OP)$; and as $PC^2 = PB^2 + BC^2 - 2BC \times BQ$, $\therefore -2AB \times PO = PC^2 - PB^2 - AB^2$. Hence we get by substituting this, $\frac{1}{4}(AP^2 + PC^2 - 2PB^2)$ as per question.



To find the square when the three lines and area of APC are given, we must draw PC, and with a radius CR = 2area ÷ AP describe an arc αRθ, through the point P, and this arc, draw the given line PA, and we have the diagonal of the square AC, from which the square is easily constructed.

SECOND SOLUTION.—*By Dr. Henry J. Anderson, N. Y.*



DEMONSTRATION. Draw PE parallel to AC. $4PAC = 4BF.FE =$ (Eucl. 13. 2), $2BF^2 + 2PF^2 - 2PB^2 =$ (Leg. 14. 3), $PA^2 + PC^2 - 2PB^2$. Q. E. D.

In the same way if PD be joined, it may be proved that $4PAC = 2PD^2 - PA^2 - PC^2$.

Cor. $PA^2 + PC^2 = PB^2 + PD^2$.

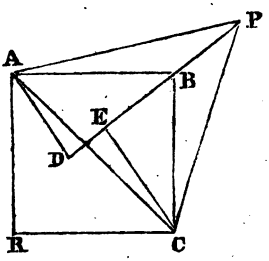
CONSTRUCTION of the problem. On PC describe a rectangle $= \frac{1}{2}(PA^2 + PC^2 - 2PB^2)$, let the side opposite to PC be produced, and with the centre P and distance PA, describe a circle cutting the produced side of the rectangle; the distance between the two points of intersection and the point C will be the diagonals of two different squares answering the conditions of the problem, in one of which the point P will be without, and in the other within the square.

THIRD SOLUTION.—*By Charles Avery, Herkimer Co. N. Y.*

It is plain that the area of the triangle APC (see the fig. to the first solution) $= \frac{1}{2}(AB^2 - AB.BO - AB.BQ) = \frac{1}{4}(AO^2 + CQ^2 - BO^2 - BQ^2 = \frac{1}{4}(AP^2 + PC^2 - 2PB^2)$.

When the three distances are given, it is but a particular case of Prop. 54. Simpson's Alg page 369, in which he shows how to construct any four-sided figure when the three distances above-mentioned are given.

FOURTH SOLUTION.—*By Mathetus, Bucks Co. Penn.*



Let ABCR be a square, and P a point without it, from which let PA, PB and PC be drawn. Join AC and produce PB, on which let fall the perpendiculars AD and CE. Now the angles ABD and CBE being complementary to each other, and also CBE and BCE, it follows that the triangles ABD and CBE are similar; and having equal hypothenuses AB and BC, their other homologous sides must be equal. Hence AD=BE, and BD=CE. By Euc. 2nd. 12. $PA^2=PB+BA^2+2PB \cdot BD$, and $PC^2=PB^2+BC^2+2PB \cdot BE$. Also by the addition of equal quantities $PA^2+PC^2=2PB^2+BA^2+BC^2+2PB \cdot BD+2PB \cdot BE$ But $BA^2+BC^2=AC^2$ and $2PB \cdot BD+2PB-BE=2PB \cdot CE+2PB \cdot AD=4APB+4PBC=4APCB$. We have also $AC^2=4ABC$. Hence $BA^2+BC^2+2PB \cdot BD+2PB \cdot BE=4APC$. Hence the original expression is reduced to $PA^2+PC^2=2PB^2+4APC$, and $APC=\frac{1}{4}(PA^2+PC^2-2PB^2)$. In order to construct the figure, we have $\frac{1}{2} AP \cdot PC+\sin. \angle APC=\frac{1}{4}(PA^2+PC^2-2PB^2)$, because $\frac{1}{2} AP \cdot PC+\sin. \angle APC$ is equal to the area of the triangle APC; $\therefore \sin APC = \frac{PA^2+PC^2-2PB^2}{2AP \cdot PC}$. Hence, as the angle APC is given, also AP and PC, the diagonal of the square, AC, is readily found, and consequently, having the diagonal, the square is easily constructed.

QUESTION XIII. (81.)—*By Mathetus, Bucks Co. Penn.*

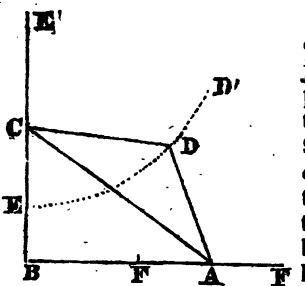
In order to determine the altitude of an inaccessible object, I measured two lines, 3 and 4, forming a right angle, and from the three angular points, ascertained the angles subtended by the altitude of the object 30°, 35°, and 40°, respectively: a solution is required.

FIRST SOLUTION.—*By Charles Farquhar, Alex. D. C.*

The ratio of the distances from the three given points to the foot of the object, is evidently given, being as the

cotangents of the angles of elevation; hence, by Prob. 56 Simpson's Exercises, those three distances are readily determined; put one of them=a, and m=the corresponding tangent of elevation; the height required=am.

SECOND SOLUTION.—*By Dr. Bowditch, Boston.*



Let A, B, C, be the points of observation, D' the object, D its projection on the plane ABC. Then per question, AB=4, BC=3, $\angle$ABC=90°. Let a=40°, b=35°, c=30°, being the several altitudes at A, B, C, respectively; and if the required height DD'=x, we shall have in the right-angled triangles, DD'A, DD'B, DD'C, the following equations,

AD=x, cotan. a.
BD=x, cotan. b.
CD=x, cotan. c.

Hence $\frac{CD}{BD}=\frac{\cot.c}{\cot.b}$; hence from a well known property of the circle, the point D will be situated on the circumference of a circle EDE', whose diameter EE', is found by continuing the line BC, and taking the points EE' so that $\frac{CE}{BE}=\frac{CE'}{BE'}=\frac{\cot.c}{\cot.b}$. In like manner, since $\frac{BD}{AD}=\frac{\cot.b}{\cot.a}$; if we continue the line BA and take on it the points F, F', such that $\frac{BF}{AF}=\frac{BF'}{AF'}=\frac{\cot.b}{\cot.a}$, the point D will be on the circumference of a circle described about the diameter FF'. Therefore these two circles being drawn, will intersect each other in *two* points D, answering the question. Having thus obtained CD, BD, AD, we get x=CD to C, &c.

THIRD SOLUTION.—*By Professor Adrain, Rutger's Col.*

Let x=altitude, t, t', t'', the cotangents of the given angles, p, p', p'', the three distances from the angular

points of the horizontal triangle to the foot of the perpendicular x; then we have

$tx=p$, $t'x=p'$, $t''x=p''$, whence by division,

$\frac{t}{t'}=\frac{p}{p'}$, $\frac{t'}{t''}=\frac{p'}{p''}$; thus the ratios of p, p', p'', are given.

The problem is therefore reduced to that of finding in a given triangle of which the sides are 3, 4, 5, a point, the distances of which from the three angular points of the triangle, shall have the given ratios of t, t', t'', which problem is constructed and calculated, in Prob. 53. Appendix to Simpson's Algebra.

It was observed by Messrs. F. M. Noll, Harlæm, and Old Numscull, N. Y. that there are two solutions to this question given in Davis's Mathematical Companion for 1816. It is numbered 314, was proposed by Mr. John Baines, and solved by Messrs. Wm. Snipe and J. Jones.

It was also very properly remarked by N'importe qui, that a solution to this question may be found in Leslie's Geometry, Prob. xxvii. page 275.

For the solution of this question, Dr. Anderson also refers to Prob. liii. Simpson's Algebra; the cotangents of the angles of observation in this question are in the same ratio of the three lines of that Problem.

A solution to this question may be also found in Ingram's Mensuration, Prob. xv. page 189.

QUESTION XIV. (82.)—*By Mr. Elias Lynch, New-York.*

Integrate $c.\frac{dx}{dy}=\sqrt{(c^2+4y^2)}$.

FIRST SOLUTION.—*By Mr. J. Ingersoll Bowditch, Boston.*

Given $c\frac{dz}{dy}=\sqrt{(c^2+4y^2)}$, $\therefore cdz=dy\sqrt{(c^2+4y^2)}$, make $\sqrt{(c^2+4y^2)}=x$, or $c^2+4y^2=x^2$. $\therefore y=\frac{\sqrt{x^2-c^2}}{2}$, and $dy=\frac{xdx}{2\sqrt{(x^2-c^2)}}$; substituting this in the equation $cdz=dy.x$, we get $cdz=\frac{x^2dx}{2\sqrt{x^2-c^2}}$. $\therefore cz=\int\frac{x^2dx}{2\sqrt{(x^2-c^2)}}=\frac{1}{2}\left(\frac{x}{2}\sqrt{x^2-c^2}+\frac{c^2}{2}\int\frac{dx}{\sqrt{(x^2-c^2)}}\right)$. But $\int\frac{dx}{\sqrt{(x^2-c^2)}}=$ log. $(x+\sqrt{(x^2$

$-c^2$). Hence $cz=\frac{x}{4}\sqrt{(x^2-c^2)}+\frac{c^2}{4}\left(\text{log. }(x+\sqrt{x^2-c^2})\right)$. Putting in this the value of x, we get $cz=\frac{\sqrt{(c^2+4y^2)}}{4}\times 2y+\frac{c^2}{4}\left\{\text{log. }(\sqrt{c^2+4y^2}+2y)\right\}=\frac{y\sqrt{(c^2+4y^2)}}{2}+\frac{c^2}{4}\left\{\text{log. }\sqrt{(c^2+4y^2)}+2y)\sqrt{}\right\}$.

SECOND SOLUTION.—*By a Correspondent, Lexington, Ky.*

Let $x=2y$; and the proposed fluxion becomes $2cz=\dot{x}\sqrt{(c^2+x^2)}=\frac{\dot{x}\times(c^2+x^2)}{\sqrt{(c^2+x^2)}}=\frac{c^2\dot{x}}{\sqrt{(c^2+x^2)}}+\frac{x^2\dot{x}}{\sqrt{(c^2+x^2)}}$. Now, the fluent of the first of these quantities is $c^2\times$h.l. $(x+\sqrt{(c^2+x^2)})$, and the fluent of the second is $\frac{1}{2}x\sqrt{(c^2+x^2)}-\frac{1}{2}c^2\times$h.l. $(x+\sqrt{(c^2+x^2)})$; see Vince's Flux. Prop. 61. Hence, the whole fluent is, $z=\frac{1}{2}c\times$h.l.$\sqrt{(x+\sqrt{(c^2+x^2)})}+\frac{x}{4c}.\sqrt{c^2+x}+$c.

THIRD SOLUTION.—*By Mr. Farrand & Benedict, Montezuma, New-York.*

If we assume $c^2+4y^2=(2y+v)^2$, we shall have $\sqrt{(c^2+4y^2)}=2y+v=\frac{c^2-v^2}{2v}+v=\frac{c^2+v^2}{2v}$. Also, $dy=-\frac{dv}{4}$; therefore $c.\frac{dz}{dy}=-\frac{4cdz}{dv}=\frac{c^2+v^2}{2v}$, and $4dz=-\frac{c^2dv}{2v}-\frac{1}{2}vdv$. Integrating we have, $4cz=-\frac{v^2}{4}-\frac{c^2}{2}$h.l. $v+$cor. reversing our substitutions, we have $4cz=-\frac{1}{4}.(\sqrt{(c^2+4y^2)}-2y)^2-\frac{1}{2}$h.l. $(\sqrt{(c^2+4y^2)}-2y)+$c.

FOURTH SOLUTION.—*By Mr. Charles Farquhar, Alex. D.C.*

We have $dz=\left(1+\frac{4}{c^2}y^2\right)^{\frac{1}{2}}dy=\left(\text{putting }\frac{2}{c}=a\right)(1+a^2y^2)^{\frac{1}{2}}dy$, which is easily integrated. $z=\int(1+a^2y^2)^{\frac{1}{2}}dy=\frac{y\sqrt{(1+a^2y^2)}}{2}+\frac{1}{2a}\text{log.}\left\{ay+\sqrt{(1+a^2y^2)}\right\}$.

FIFTH SOLUTION.—*By Mr. James Maccully, N. Y.*

If we compare the given equation $c\frac{dz}{dy} = \sqrt{(c^2 + 4y^2)}$ with Emerson's Fluxions, page 244, Ex. 12, we have $c^2 = r^4$, $c = r^2$, and $4 = c^2$; hence by substitution, $r^2\frac{dz}{dy} = \sqrt{(r^4 + c^2y^2)}$, whose integral corrected is $\frac{y}{r^2}\sqrt{(r^4 + c^2y^2)} + \frac{2.302585r^2}{2} \times \log.\left(\frac{cy + \sqrt{(r^4 + c^2y^2)}}{r^2}\right)$

It has been very properly observed by several of our Contributors, that the integral of this differential equation may be found in several mathematical works.

Doctor Adrain remarks, that the integral given in Tab. 34, Hirsch's Tables, is $cs = \frac{1}{2}y\sqrt{(c^2 + y^2)} + \frac{c^2}{4}$ h. l. $(2y + \sqrt{(c^2 + 4y^2)}) + c$. Professor Matthews also observes, that the integral is given as above, by the 25th form of the Table of fluents, Barlow's Mathematical and Philosophical Dictionary. Professor Strong also remarks that this equation is integrated in Vince's Fluxions, Prob. xxiv. Ex. 3. For, let $b = \frac{c}{2}$; then, the given equation becomes $ds = \frac{(y^2 + a^2) \times dy}{b}$, which is the identical equation given by Vince. It has been also observed by Dr. Anderson, and by Mr. Charles Avary, that the integral of this equation is evidently the arc of the common Parabola, see Vince's Fluxions, page 85.

QUESTION XV. (83.)—*By Mr. Elias Lynch, N. Y.*

To find the equation and area of a curve, whose subtangent $= \frac{2x(a-x)}{3a-2x}$.

FIRST SOLUTION.—*By Professor Matthews, Lex. Ken.*

The question requires that $\frac{2x(a-x)}{3a-2x} = \frac{ydx}{dy}$, whence $\frac{dy}{y} = \frac{(3a-2x)dx}{2x(a-x)} = \frac{Adx}{2x} + \frac{Bdx}{a-x}$ by assumption, from which by reducing to a common denominator, and bringing all the terms of the numerators to the same side of the equation, we obtain $Aa - 3a + (2 + 2B - A)x = 0$, whence $A = 3$ and $B = \frac{1}{2}$; consequently $\frac{dy}{y} = \frac{3dx}{2x} + \frac{dx}{2(a-x)}$; now

$$\int \cdot \frac{3dx}{2x} + \int \cdot \frac{dx}{2(a-x)} = \int \cdot \left(\tfrac{3}{2} \cdot \frac{dx}{x}\right) + \int \cdot \left(-\tfrac{1}{2} \cdot \frac{-dx}{a-x}\right)$$

$=$ hyp. log. $x^{\frac{3}{2}}$ $-$ hyp. log. $(a-x)^{\frac{1}{2}} =$ hyp. log. $\dfrac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}$ $=$ hyp. log. y; therefore $y = \dfrac{x^{\frac{3}{2}}}{(x-x)^{\frac{1}{2}}}$ or $y^2 = \dfrac{x^3}{a-x}$, the equation of the curve. To obtain the area we have to integrate $\dfrac{x^{\frac{3}{2}}dx}{(a-x)^{\frac{1}{2}}} = \dfrac{x^2 dx}{(ax-x^2)^{\frac{1}{2}}}$, the integral of which by the method of continuation (vide Hutton's Mathematics, vol. 2. p. 350) is $\frac{3}{4}\,a \times$ cir. arc to rad. $\frac{1}{2}\,a$, and vers. $x - \dfrac{\frac{3}{2}a+x}{2}\sqrt{(ax-x^2)}+$ const.

Second Solution.—*By Mr. Joseph C. Strode, Strodesville, Chester Co. Penn.*

$\dfrac{2x\,(a-x)}{3a-2x} = \dfrac{ax-x^2}{\frac{3}{2}a-x} = \dfrac{ydx}{dy}$; whence $\dfrac{\frac{3}{2}adx-xdx}{ax-x^2} = \dfrac{dy}{y}$. Assume $q \times$ hyp. log. $(ax^r - x^{r-1})$ for the integral of the first part of the equation. Its differential is $q \times$

$$\frac{rax^{r-1}dx - \overline{r+1} \times x^r dx}{ax^r - x^{r+1}} = \frac{qardx - q \times \overline{r+1} \times xdx}{ax-x^2} = \frac{\frac{3}{2}adx - xdx}{ax-x^2}.$$

Whence by collating the homologous terms, $qar = \frac{3}{2}\,a$, $q \times \overline{r+1} = 1$; hence $r = -3$, $q = -\frac{1}{2}$. Therefore the integral $= -\frac{1}{2} \times$ hyp. log. $(ax^{-3} - x^{-2}) =$ hyp. log. y or $y = (ax^{-3} - x^{-2})^{-\frac{1}{2}} = \dfrac{1}{(ax^{-3}-x^{-2})^{\frac{1}{2}}}$ $\therefore y^2 = \dfrac{1}{ax^{-3}-x^{-2}} = \dfrac{x^3}{a-x}$ which is the equation of the Cissoid of Diocles, whose area (Art 125, Simpson's Flux.) is $x^2 \sqrt{\dfrac{x}{a}} \times \dfrac{2}{5} + \dfrac{x}{7a} + \dfrac{x^2}{12a^2} + \dfrac{5x^3}{88a^3}$.

Third Solution.—*By Mr Charles Wilder, Baltimore, Maryland.*

$\dfrac{ydx}{dy} = \dfrac{2x\,(a-x)}{3\,a-2x}$ by the question, or $\dfrac{dy}{y} = \dfrac{dx\,(3a-2x}{2x\,(a-x)}$ $= \dfrac{3dx}{2x} + \dfrac{dx}{2\,(a-x)}$, and integrating log. $y = \frac{3}{2}$ log. $x - \frac{1}{2}$ log. $(a-x)$, or $y = \dfrac{x^3}{a-x}$, the equation to the Cissoid of Diocles.

Hence the area may be readily found by integrating the equation $ydx = \frac{x^{\frac{3}{2}}dx}{(a-x)^{\frac{3}{4}}}$.

Professors Adrain, and Anderson, in their solutions to this question which are similar to the above, observe that the general integral is log. cy^2 = log. x^3 — log $(a-x)$ or $cy^2 = \frac{x^3}{a-x}$. The curve expressed by this equation comprehends the common Cissoid as a particular case, namely, by taking the arbitrary constant c equal to unity.

FOURTH SOLUTION.—*By a Correspondent, Lex. Ken.*

Let y be the ordinate, and we have by the question $\frac{\dot{y}x}{\dot{y}} = \frac{2x \times \overline{a-x}}{3a-2x}$; and consequently $\frac{2\dot{y}}{y} = \frac{a-x}{x^3} \times \frac{3ax^2\dot{x} - 2x^3\dot{x}}{\overline{a-x}^2}$.

Hence the fluent, 2 log. y = log. $\frac{x^3}{a-x}$, or, by the nature of logarithms, $y^2 = \frac{x^3}{a-x}$, is the equation required; and the curve is therefore the Cissoid of Diocles.

Multiply the equation by $\dot{x}$ and we have $y\dot{x} = \dot{A} = \overline{a-x}^{-\frac{1}{2}} \times x^{\frac{3}{2}}\dot{x}$; and this, by expanding $\overline{a-x}^{-\frac{1}{2}}$ and taking the fluent of each term, will give the

$$\text{area} = \sqrt{\frac{x^5}{a}} \times \left(\frac{2}{5} + \frac{2x}{14a} + \frac{6x^2}{72a^2} + \text{\&c.}\right)$$

This question was solved in a similar manner by Mr. James Maccully.

Old Numscull, N. Y. remarked that, "having found the equation, $y^2 = \frac{x^3}{a-x}$, of the curve which is the Cissoid of Diocles, the area is given in Dealtry's Fluxions, Second Ed. Chap. 16. Prob. 5, equal to $\frac{3a}{2}$ × arc of a semicircle = 3 generating circles."

QUESTION XVI. (84.)—*By Mr. Charles Wilder, Baltimore.*

Integrate $\frac{2n\phi y dy}{(1 \pm ny^2)}$, y being the sine of the arc ϕ.

FIRST SOLUTION.—*By Professor Strong, Hamilton College.*

If the sign — be used, put $\frac{\phi}{1-ny^2} = z$, then by differentiation, I have $\frac{2n\phi y dy}{(1-ny^2)^2} + \frac{d\phi}{1-ny^2} = dz$ ∴ $\frac{2n\phi y dy}{(1-ny^2)^2} = dz -$

$\frac{d\varphi}{1-n\sin.^2\varphi}$; hence by integration $\int\frac{2n\phi y dy}{(1-ny^2)^2}=z-\int\frac{d\varphi}{1-n\sin^2\phi}$, put $\tan.\phi=x$, then $\frac{d\phi}{\cos\varphi^2}=dx$, and we have $d\varphi=\cos\varphi^2 dx$, but $\sin.\phi=\frac{x^2}{1+x^2}$, and our expression reduces to $\int\frac{2n\phi y dy}{(1-ny^2)^2}=z-\int\frac{dx}{1+(1-n)x^2}=z-\frac{\theta}{\sqrt{(1-n)}}$; or it $=z-\frac{1}{2\sqrt{(n-1)}}\times$h.l.$\frac{1+x\sqrt{(n-1)}}{1-x\sqrt{(n-1)}}$ according as $n\angle 1$ or $n>1$, in which $\theta=$arc rad. (1) tan. $x\sqrt{(1-n)}$ and, the integral needs no correction if it commences when $\phi=0$. But if the sign $+$ be used, then put $\frac{\varphi}{1+ny^2}=z'$, and we have as before $-\frac{2nydy\varphi}{(1+ny^2)^2}+\frac{d\phi}{1+ny^2}=dz'$; hence, $\int\frac{2nydy\varphi}{(1+ny^2)^2}=\int\frac{d\phi}{1+n\sin.^2\varphi}dz'$ put $x=\tan.\phi$ as above, and it reduces to $\int\frac{2nydy\varphi}{(1+ny^2)^2}=\int\frac{dx}{1+(1+n)x^2}-dz'=\frac{\theta'}{\sqrt{1+n}}-z'$; θ'arc $=$ rad. (1) tan. $=x\sqrt{(1+n)}$; this needs no correction if the integral commences when $\varphi=0$. This question as appears reduces by the above process immediately to the question which I proposed in No. 4 of the Diary.

Second Solution.—*By Mr. J. Ingersoll Bowditch, Boston.*

Put $1+ny^2=z\therefore ny^2=z-1\therefore 2nydy=dz$. Hence by substitution we get $\varphi dz z^{-2}=-z^{-1}\varphi+\int z^{-1}d\phi=-\frac{\varphi}{z}+\int\frac{d\varphi}{1+ny^2}$; and as $d\varphi=\frac{d\sin.\theta}{\cos.\varphi}=\frac{dy}{\sqrt{(1+y^2)}}\therefore$ we have $-\frac{\varphi}{1+ny^2}+\int\frac{dy}{(1+ny^2)\sqrt{(1+y^2)}}$. The integral of the last term is found by the tables to be $=\frac{1}{\sqrt{(\mp n-1)}}$ log.

$\frac{\sqrt{(1-y^2)}+y\sqrt{(\mp n-1)}}{\sqrt{(1+ny^2)}}$, and $\sin.^{-1} y\sqrt{\left(\frac{\pm n+1}{1\pm ny^2}\right)}$. Hence the answers will be $-\frac{\varphi}{1+ny^2}+\frac{1}{\sqrt{(\mp n-1)}}\text{—lgg.}\frac{\sqrt{(1-y^2)}+y(\mp n-1)}{\sqrt{(1+ny^2)}}$ and $-\frac{\varphi}{1+ny^2}+\sin.^{-1} y\sqrt{\frac{\pm n+1}{1\pm ny^2}}$.

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THIRD SOLUTION.—*By Mr. John H. Willets, Philadelphia.*

In order to determine $\int\frac{2n\varphi y dy}{(1\pm ny^2)^2}$, take $d.\frac{\phi}{(1\pm ny^2)}$ $=\frac{(1\pm ny^2)d\phi\pm 2n\phi y dy}{(1\pm yn^2)^2}=\frac{d\phi}{1\pm ny^2}\pm\frac{2n\phi y dy}{(1\pm ny^2)^2}$, therefore $\pm\int\frac{2n\phi y dy}{(1\pm ny^2)^2}=\int\frac{d\phi}{1\pm ny^2}-\frac{\varphi}{1\pm ny^2}$; but since $y=$ the sine φ; we have $d\varphi=\frac{dy}{\sqrt{(1-y^2)}}$; therefore $\int\frac{d\varphi}{1\pm ny^2}=\int\frac{dy}{\sqrt{(1-y^2)}\times(1\pm ny^2)}$. Now, if we assume $y=\frac{1}{\sqrt{(z^2+1)}}$, we shall have $dy=-\frac{zdz}{(z^2+1)^{\frac{3}{2}}}$, and $\sqrt{(1-y^2)}=\frac{z}{\sqrt{z^2+1}}$; and $1\pm ny^2=\frac{z^2+1\pm n}{z^2+1}$. Consequently $\int\frac{dy}{(1\pm ny^2)\cdot\sqrt{1-y^2}}$ $=-\int\frac{dz}{z^2+1\pm n}=\frac{-A}{1\pm n}$, where A=Circ. arc. (rd. $\sqrt{(1\pm n)}$ and tang. $z\left(=\frac{1}{y}\sqrt{(1-y^2)}\right)$.

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FOURTH SOLUTION.—*By Mr. Charles Avary, Herkimer County.*

Put $\frac{\phi}{1\pm ny^2}=p$, then $\mp\frac{2n\varphi y dy}{(1\pm ny^2)^2}=p-\frac{d\phi}{1\pm ny^2}$, we have only to integrate $\frac{d\phi}{1\pm ny^2}$, which has been done in No. V. of the Diary. Put $x=$tan. φ, and $d\phi=dx.\cos\varphi^2$

$= \frac{dx}{1+x^2}$; $\therefore \frac{d\phi}{1 \pm ny^2} = \frac{dx}{1+(1 \pm n)x^2} =$ arc $\left(\text{tan. } x \text{ and radius } \frac{1}{(1 \pm n)^{\frac{1}{2}}}\right)$.

FIFTH SOLUTION.—*By Professor Matthews.*

The given differential, integrated on the supposition of ϕ being constant, gives $-\frac{\phi}{1 \pm ny^2}$, the differential of which, making both y and ϕ vary, is $-\frac{d\phi}{1 \pm ny^2} \pm \frac{2ny\phi dy}{(1+ny^2)^2}$; this subtracted from the given differential, or added thereto according to the sign, gives a difference or sum $= \pm \frac{d\varphi}{1 \pm n \sin.^2 \varphi}$, the integral of which by Dr. Bowditch's solution to question 14 last number, is $\pm \frac{2}{\sqrt{(a^2-b^2)}} \times$ arc $\left(\text{tan.} = \sqrt{\frac{a-b}{a+b}} \text{ tan. } \phi\right)$ + cons. therefore the whole integral is $\pm \frac{2}{\sqrt{(a^2-b^2)}}$ arc. $\left(\text{tan.} = \sqrt{\frac{a-b}{a+b}} \text{tan.} \varphi\right) - \frac{\phi}{1 \pm n \sin.^2 \varphi} + c.$

QUESTION XVII. (85.)—*By Professor Strong, Hamilton College.*

Investigate the motion of the apsides in orbits differing infinitely little from circles, in a more concise manner than Newton has done in his ninth section, supposing that there is any number whatever of disturbing forces, and whether they are centrifugal or centripetal; the disturbing force being always supposed to act in the direction of the radius vector.

FIRST SOLUTION.—*By Professor Adrain, Rutger's College, N. B.*

Let r = the radius vector of the two bodies which is common to the fixed and revolving orbits; c and c' the two areas described in an unity of time, ϕ and ϕ' the corresponding forces at the same distance r, and t = the time of the motion.

By the general theory of central forces we have

$$\frac{ddr}{dt^2} = \frac{c^2}{r^3} - \varphi,$$

and $$\frac{ddr}{dt^2} = \frac{c'^2}{r^3} - \phi',$$

whence $$\frac{c^2}{r^3} - \varphi = \frac{c'^2}{r^3} - \phi'$$

and therefore $$\phi' - \phi = \frac{c'^2 - c^2}{r^3},$$

which is the difference between the forces necessary for the description of the two orbits in the same time.

If the fixed orbit be an ellipse, of which the parameter $=p$, and focus the centre of force, we have

$$\varphi = \frac{c^2}{pr^2}, \text{ therefore } \phi' = \frac{c^2}{pr^2} + \frac{c'^2 - c^2}{r^3}.$$

To apply this to the motion of the apsides of an orbit, let $\frac{fr}{r^3}$ be the force with which the body is urged to the centre of force in the moveable orbit, fr denoting any function of r, then the two forces ϕ' and $\frac{fr}{r^3}$ may be made to coincide indefinitely near when the orbit is indefinitely near to a circle.

Since $\frac{c^2}{pr^2} + \frac{c'^2 - c^2}{r^3} = \frac{fr}{r^3}$ indefinitely near, we have $c^2 r + p(c'^2 - c^2) = pfr$, (A), which by taking $p = r$, gives $c'^2 = fr$.

Again, by taking the differential of equation (A), by making r vary, we have $c^2 = pf'r$; in which, putting r for p, we have $c^2 = rf'r$.

Now, by division we have $\frac{c'^2}{c^2} = \frac{fr}{rf'r}$, and therefore $\pi \frac{c'}{c}$ $= \pi \sqrt{\frac{fr}{rf'r}}$; which expresses the angle contained between the higher and lower apsis of the fixed curve described by the ball moving in the revolving orbit.

Ex. 1. Let $\frac{fr}{r^3} = A$, then $fr = Ar^3$, $f'r = 3Ar^2$, and $\pi \sqrt{\frac{fr}{rf'r}} = \frac{\pi}{\sqrt{3}}$, or $= \frac{180^\circ}{\sqrt{3}}$ in degrees.

Ex. 2. Let $\frac{fr}{r^3} = Ar$; here $f'r = Ar^4$, and $f'r = 4Ar^3$; whence $\pi \sqrt{\frac{fr}{rf'r}} = \frac{\pi}{2}$.

Ex. 3. Let $\frac{fr}{r^3} = Ar^n$, then $fr = Ar^{n+3}$, ; hence $f'r = (n+3) \cdot Ar^{n+2}$, and therefore $\pi \sqrt{\frac{fr}{rf'r}} = \frac{\pi}{\sqrt{(n+3)}}$.

This expression $\frac{fr}{rf'r}$, will always be found in an absolute number without the numerical length of r or any other line, when the value of fr is properly given in lines and ratios.

This problem has been resolved by several mathematicians, as Hermann in his Phoronomia, Euler in his Mechanics, Simpson in his Essays, Wheewell in his Dynamics, &c.

—

Second Solution.—*By N. Bowditch, L. L. D. Boston.*

Using the symbols of La Place's Mec. Cel. T. I. p, 113, namely x, y, the rectangular co-ordinates of the planet, the sun being the origin, r the radius vector, v the angle described about the sun, central force φ, t the time, cdt= twice the area described about the sun in the time dt. Then by that page of La Place's work, the central force φ is

$$\varphi = \frac{c^2}{r^3} - \frac{c^2}{2} \cdot \frac{d. \left\{ \frac{dr^2}{r^4 dv^2} \right\}}{dr}.$$

Now in Newton's method, it is supposed that the body in the moveable plane describes equal areas in equal times, but greater or less than cdt ; we shall therefore represent the corresponding area in this hypothesis by cdt, and the angle cdv will become cdv in the moveable orbit, and the force ϕ in this new orbit being called φ', we shall obtain its value by changing in the preceding value of φ, c into c, dv into $\frac{cdv}{c}$, r and dr being the same ; hence

$$\varphi' = \frac{c^2}{r^3} - \frac{c^2}{2} \cdot \frac{d. \left\{ \frac{dr^2}{r^4 dv^2} \cdot \frac{c^2}{c^2} \right\}}{dr} = \frac{c^2}{r^3} - \frac{c^2}{2} \cdot \frac{d. \left\{ \frac{dr^2}{r^4 dv^2} \right\}}{dr};$$

subtracting from this the above value of φ, we get

$$\phi' - \varphi = \frac{c^2 - c^2}{r^3}, \text{ or } \phi' = \phi + \frac{c^2 - c^2}{r^3},$$

which is the same as Newton's rule. For by page 114 of La Place, T. I. $\varphi = \frac{c^2}{a(1-e^2)} \cdot \frac{1}{r^2} = \frac{c^2}{R} \cdot \frac{1}{r^2}$, e being the eccentricity, and 2R the *latus rectum*. Moreover, if we put $c^2 = F^2R$, $c^2 = G^2R$, $r = A$, we shall get $\varphi' = \frac{F^2}{A^2} + \frac{RG^2 - RF^2}{A^3}$, exactly as Newton, Prop. xliv. Lib. 1.

QUESTION XVIII. (86.)—**OR PRIZE QUESTION.**
By Mr. J. H. Swale, Liverpool, England.

A right line is given in position: to assign the position of a point in the periphery of a circle (given in position and magnitude) such, that thence drawing given and equal tangents, and demitting perpendiculars (upon the line given in position) from the extremities of the tangents; the rectangle of those perpendiculars shall be given, or a maximum or minimum.

THE PRIZE SOLUTION.—*By Professor Adrain.*

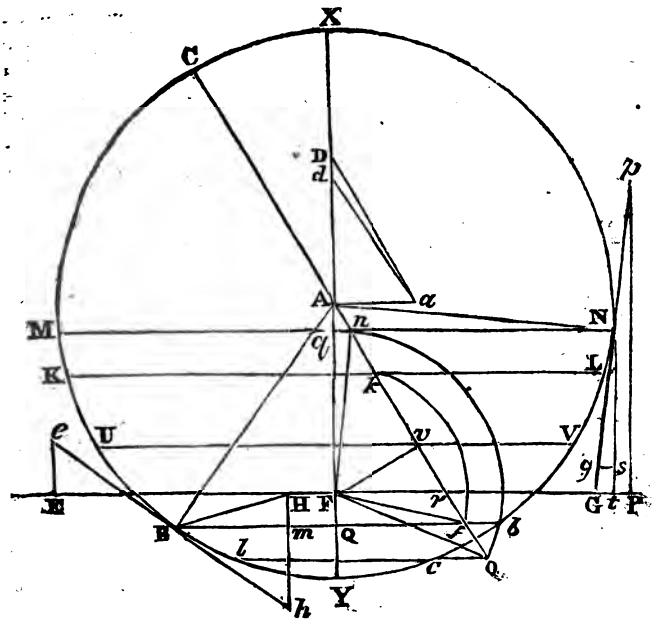
Let A be the centre of the given circle BbC, and EG the straight line given in position, perpendicular and parallel to which draw DAF and A*a*, making AD equal to one of the given equal tangents, and A*a* the side of a square equal to the given rectangle.

Join a*D*, and make *ad* equal to AD. In EG take F*r* a fourth proportional to the given radius AB, AD, AF, and join A*r*, and with the centre F and radii A*d*, a*D* describe the circular arcs *fk*, *no*, meeting the straight line A*r*, produced both ways if necessary, in *f*, *k*, *n*, *o*; through which points draw the four straight lines B*b*, KL, MN, *cl*, parallel to EG and meeting the circumference of the given circle in B, *b*, *l*, *c*, K, L, M, N, the eight points required.

To demonstrate thus far the points B, *b*, K, L, join AB, F*f*; draw E*bh* at right angles to AB, making B*e* and B*h* each equal to AD, on EG let fall the perpendiculars E*e*, *h*H, join BH, and let AF, *h*H meet B*b* in Q and *m*.

The triangles AF*r*, AQ*f* are similar, as also ABQ, B*hm*; therefore AQ is to Q*f* as AF to F*r*; that is, by construction as AB to B*h*, that is by sim. tri. as AQ to B*m*, and therefore B*m* is equal to Q*f*, and consequently BH is equal to F*f* or A*d*.

Now, since B*h* is equal to B*e*, it is evident that *hm* is half the sum, H*m* half the difference of the perpendiculars



eE, *h*H, and therefore the rectangle eE, *h*H, is equal to the difference of the squares of *hm*, *m*H ; that is, to the difference of the squares of B*h*, BH, that is, of *ad*, *d*A, which is the given square of A*a*. The demonstration is similar for the points *b*, K, and L.

To demonstrate the construction for the points *c*, *l*, M, N, join AN, and draw *p*N*g*, at right angles to it, making N*p*, N*g* each equal to AD ; from *p*, N, *g* draw *p*P, N*t*, *g*G at right angles to EG, and *gs* parallel to the same, and let MN meet AF in *q*.

Now, as in the preceding demonstration we prove, that *nq* is equal to *gs* or G*t*, and consequently NG is equal to F*n* or *a*D.

And since, as in the preceding case, the rectangle *p*P, *g*G is equal to the difference of the squares of N*t*, N*s*, that is, to the difference of the squares of NG, N*g* or of *a*D, AD, therefore the rectangle *p*P . *g*G is equal to the given square of A*a*. A similar proof is applicable to the points M, *c*, *l*.

When the perpendiculars e*k*, *h*H are on opposite sides

of EG, it is evident that Aa will be a maximum when Ad or Ff is a minimum; that is, when Ff coincides with Fv at right angles to Ar, and therefore a straight line vv drawn through v parallel to EG, will cut the circumference of the given circle in v and v, the two points which correspond to the required maximum. Again, when the perpendiculars are on the same side of EG, it is plain that Aa will be a maximum or minimum when AD is so; that is, when Fn or Fo is a max. or min. that is, when the points N and c coincide with the points x and y the extremities of the diameter which is perpendicular to the given straight line EG. When the straight line EG meets the given circle, the rectangle of the perpendiculars on the same side of the given line evidently admits of two maxima, viz. at x and y; but when EG does not meet the circle, the rectangle is a maximum at the former of these points, and a maximum or minimum at the latter.

Lastly, when some of the parallels through f, k, n, o, do not meet the circle, there will be fewer than eight points that can satisfy the condition of a given rectangle, and if none of the parallels meet the circle, the problem becomes impossible.

Analytical Solution to the Prize Question by the Same.

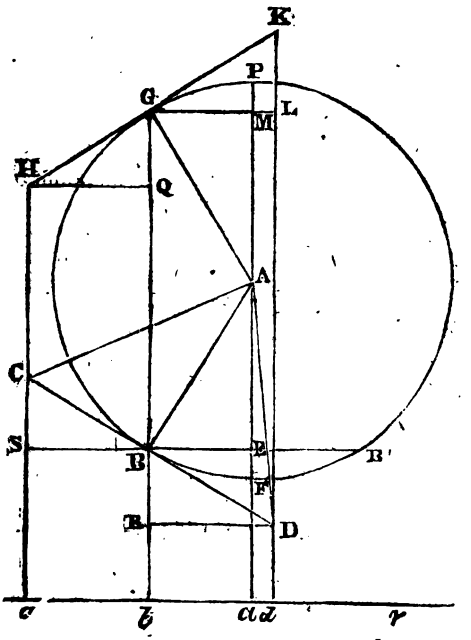
Let A be the centre of the given circle FBG, cr the given straight line, CBD the given tangent, in which BC is equal to BD, the point of contact being B.

From A, B, C, D, let fall on cr the perpendiculars Aa Bb, Cc, Dd, and through B and D draw ES, DR parallel to cr.

Put a = the given perpendicular Aa,
b = the given tangent BC or BD,
r = the radius AB, or AF,
f = the distance AD,
x = BE, y = AE, v = FE.

and let z = the rectangle $Cc \cdot Dd$, which is given, or is a max. or mim.

By sim. tri. ABC, BCS, BDR, $r : x :: b : \frac{bx}{r}$ = CS = BR, and $Bb = a - y$; therefore $Co = a - y + \frac{bx}{r}$, $Dd = a - y - \frac{bx}{r}$, consequently we have $Cc \cdot Dd = \left(a - y + \frac{bx}{r}\right) \cdot \left(a - y - \frac{bx}{r}\right)$; that is, $z = (a-y)^2 - \frac{b^2x^2}{r^2}$, in which putting $r^2 =$



y^2 for x^2, we have $z=a^2-b^2-2ay+y^2+\frac{b^2y^2}{r^2}$, that is, $z=a^2-b^2-2ay+\frac{f^2y^2}{r^2}$.

From this quadratic equation when z is given, the value of y, when possible may always be found. But to find the limits of possibility let us find the maximum and minimum values of z.

Taking the differential of z we have $\frac{dz}{dy}=-2a+\frac{2f^2y}{r^2}$, which equated to Zero, gives $y=\frac{ar^2}{f^2}$, or $\frac{y}{r}=\frac{ar}{f^2}$.

This equation which is to determine the max. or min. is possible if ar be less than f^2, or equal to it, because $\frac{y}{r}$ cannot be greater than unity, and in this case there may be a max. or min. but when $\frac{ar}{f^2}$ is greater than unity,

or ar greater than f^2, there is neither max. nor min. Supposing the value of y to be possible, we must ascertain whether y corresponds to a max. or min.

By the common rule $\frac{ddz}{dy^2} = \frac{2f^2}{r^2}$, which is necessarily positive, and therefore the point B found by this investigation, makes the rectangle cc . nd, a mininum.

There is, however, a defect in this solution, because when y becomes equal to r, that is, when E coincides with F, y cannot have a fluxion because y cannot increase at that point; or, which amounts to the same thing, $dy=o$ at F, which gives $dz=dy\times\left(-2a+\frac{f^2y}{r^2}\right)=o\times\left(-2a+\frac{f^2y}{r^2}\right)$, whence $dz=o$: but whether the equation $dz=o$ indicates a max. or min. or neither, may be determined as follows.

For this purpose substitute for y its value $r-v$, and we have

$$z=(a-r)^2+2\cdot\left(\frac{ar-f^2}{r}\right)\cdot v+\frac{f^2v^2}{r^2}.$$

This Solution will be continued in our next Number.

ACKNOWLEDGMENTS, &c.

The following gentlemen favoured the editor with solutions to the questions in Article XI No. V. The figures annexed to the names refer to the questions answered by each as numbered in this article.

Dr. Adrain, Dr. Bowditch, Dr. Anderson, Professor Strong, Charles Farquhar, and Charles Wilder, each answered all the questions.

Professor Matthews, and Charles Avary, each answered all but 17; James Macully, all but 16, 18; John H. Willets and J. Ingersoll Bowditch, each answered all but 7, 17; Joseph C. Strode; all but 8. 10. 17; William Lenhart, all but 14, 16, 17; James Divver, all but 14. 15. 16. 17.; Farrand N. Benedict, all but 7, 10, 16, 17. Selah Hammond answered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12; Solomon Wright and Mathetus, each answered 1, 2. 3, 4, 5, 6, 9, 11, 12, 13, 18; James Hamilton, N. J. 1, 2, 3, 4, 5, 6, 7, 9, 11. 18; Gerardus B. Docharty, 1, 2, 3, 4, 5, 6, 7, 9, 13; Joseph Mc Kean, 1, 2, 3, 4, 5, 6, 9, 10; Edward Giddins, 1, 2. 3, 4, 5, 6, 9, 11; N'importe qui, 1, 3, 4, 5, 6, 11, 13; William F. Kells, 1, 2, 3, 4, 5, 6, 11; Cyril Pascalis, 1, 2, 3, 4, 5, 6, 10; F. M. Noll. 1, 2, 3, 4, 5, 6, 13; D. T. Disney and William Vogdes, each answered 1, 2, 3, 4. 5, 6, 9; Mary Bond, 2, 3, 4, 5, 6, 8, 11; Michael Floy, 2, 3, 5, 6, 7, 9, 13; N. Leeds, 1, 2, 3, 4, 5, 6, 8; Al-

pheus Bixby, 1, 2, 3, 4, 5, 6, 10; Deboor V. Burger, 2, 3, 4, 5, 6, 8; Enoch Lanning, 1, 2, 3, 4, 5, 6; Daniel Shanley and James Sweeny, each answered 1, 2, 3, 4, 5, 6; James O Farrell, 1, 2, 3, 4, 5, 9; Robert Parry and Ransford Wells, each answered 1, 2, 4, 5, 6; Old Numscull, N. Y. 4, 9, 13, 15, 18; S. of Brooklyn, L. I. 2, 3, 4, 5, 6; James Elder and D. T. Disney, Cincinnati, 9, 11, 13, 18; a Correspondent, Lex. Ken. 13, 14, 15, 18; William S. Smith, 2, 3, 4, 5; William A. W. Stigleman, 2, 3, 5, 6; Juvenis, 2, 18; Dennis W. Carmody, 11; Benjamin Hallowell, 13; Messrs. Evans and Alsop, 2; and S. of New-York, 1.

The Prize has been awarded to Professor Adrain, Rutger's College, New Brunswick, N. Y.

In the list of answers published in No. V. the name of Mr. Solomon Wright was inadvertently omitted. He answered the questions numbered 1, 2, 3, 4, 5, 6, 7, 8.

We are sorry to announce the death of our old Friend and Correspondent, John Capp Esq. *of Harrisburg, Penn. which happened the 14th of February, 1826, after a protracted illness of several months.*

ARTICLE XIII.

NEW QUESTIONS,

TO BE RESOLVED BY CORRESPONDENTS IN No. VII.

QUESTION I. (87.)—*By Mr. D. T. Disney, Cincinnati.*

Given $x^3+y^3-x-y = 249740$

$xy+x+y = 8516,$

to determine x and y without substituting any other values for them.

QUESTION II. (88.)—*By Mr. Gerardus B. Dochurty, Flushing, L. I.*

Given $x^4+y^4+x^2+y^2+2x^2y^2 = 238632$

$x^4+z^4+y^2+z^2+2y^2z^2 = 1640$

$x^2+y^2+z^2 = \sqrt{(275100-x^2-y^2-z^2)},$

to find the values of x, y, and z.

QUESTION III. (89.)—*By Mr. Michael Floy, N. Y.*

Given the five following equations:

$$x+y+z+v+u = 12.15$$
$$x+y+z+vu = 9.15$$
$$x+y+z+zvu = 5.68$$
$$x+yzvu = 5.09$$
$$xyzvu = .46,$$

to determine the values of x, y, z, v and u.

QUESTION IV. (90.)—*By E. Giddins, Fort Niagara.*

The sum of five numbers in geometrical progression, and their fourth difference being given to determine the numbers by a cubic equation.

QUESTION V. (91.)—*By Mr. Selah Hammond.*

The perpendicular from the right angle of a triangle upon the hypothenuse is equal to the product of the legs, and makes the greater to the lesser segment as 3 to 2; to determine the triangle.

QUESTION VI. (92.)—*By Mr. S Wright, Bucks Co. Penn.*

In a right-angled triangle, there is given the difference between the lines bisecting the base and vertical angle; and the part of the base between those two lines, to determine the triangle.

QUESTION VII. (93.)—*By Mr. James Hamilton, Trenton.*

What is the radius of a circle, whose centre being taken in the circumference of another containing an acre, shall cut off half its contents?

QUESTION VIII. (94.)—*By Dennis W. Carmody, N. Y.*

It is required to find an arc of a given circle, such, that if a perpendicular be let fall from the extremity of the sine on the tangent, and the secant to the point where the perpendicular meets the tangent be drawn, the rectangle under the versed sine and part of the secant between the centre and the sine is to the rectangle under the cosine, and that part of the sine between the arc and secant in a given ratio as $m : n$. And show the geometrical analysis with the construction.

QUESTION IX. (95.)—*By Mr. James Phillips.**

Given the base, the vertical angle, and the line drawn from the centre of the inscribed circle bisecting the base, to determine the triangle.

QUESTION X. (96.) *By Mr. Joseph C. Strode.*

To find an arc, such that its cosine, sine and tangent, shall be in harmonical proportion.

* This gentleman has been lately appointed Professor of Mathematics, North Carolina University, Chapel Hill near Raleigh.

QUESTION XI. (97.)—*By Mr. James Divver, S. C. College, Columbia.*

If on each of the radii AB, AC, of a quadrant of a circle semicircles AEDB, AFDC, be respectively described, cutting each other in the points A, D, (the point A being the centre of the circle,) the area included by the two arcs AED, DFA, will always be equal to the area included by the two arcs CD, DB, and the quadrantal arc BC And the area included by the arcs AFD, DB and the right line AB, will be equal to the square of the right line drawn from the point D to the middle of AB.

QUESTION XII. (98.)—*By Mr. James Macully, N. Y.*

Given the vertical angle of a plane triangle, the straight line drawn from its centre of gravity to the middle of the base, and the ratio of its greatest inscribed ellipse, to that of its greatest inscribed parabola ; to determine the triangle.

QUESTION XIII. (99.)—*By Mr. Charles Farquhar, Alex. D. C.*

Find values of x and y, so that $(x^n \pm y^n)^{\frac{1}{n+1}}$ shall be rational and integral, n being any positive number whatever.

QUESTION XIV. (100.)—*By Mr. William Lenhart.*

To find when the subtangent of a Cissoid is a maximum.

QUESTION XV. (101.)—*By Mr. Farrell Ward, N. Y.*

Admitting the equation of a curve to be, $y = r \cdot \cos z$, when y is the radius vector, and z the angle which it makes with the base of the curve, it is required to determine the area of the greatest rectangle inscribed in it.

QUESTION XVI. (102.)—*By Professor Strong, Ham. Coll.*

A given right cone has its axis vertical and its base horizontal, a heavy body moves from a certain given point on the surface of the cone with a given velocity ; it is required to determine the line on the surface of the cone which it must trace out in order that its velocity estimated in the direction of the slant height may be constant.

QUESTION XVII. (103.)—*By Professor Dean, Burlington College, Vermont.*

Suppose the pressure of the atmosphere to be equal to that of a column of water 30 feet high, and a pump to be constructed with one valve at the surface, and the other to move between 10 and 11 feet above that surface, having a motion of one foot; let the valves close perfectly tight, and open with perfect ease; how high will the action of the piston raise the water in the pump.

QUESTION XVIII. (104)—*By Philotechnus, Phil.*

Emerson in his Miscellanies finds the pressure on the axis of the wheel and axle when in motion to be $\frac{2pq\,(a+b)}{ap+bq}$, in which p and q represent the power and weight, and a, b, the radii of the wheel and axle; is this expression correct?

QUESTION XIX. (105.)—*By Mr. John Rochford, Virginia.*

Suppose a heavy body placed on the vertex of an inclined plane, and left at liberty to descend freely at the very instant that the lower end of the plane begins to be moved uniformly in a horizontal direction towards the altitude; required the time of descent to the horizon, velocity acquired, and nature of the curve described, when the plane slides continually on the vertical point at a given height above the horizon; the horizontal motion being given.

QUESTION XX. (106.)—*By Prof. Anderson, Col. Coll. N. Y.*

Required the length and position of the principal axes or parameters of the surface, which is the *locus* of the centres of all the spheres which touch two straight lines given in position any way in space.

QUESTION XXI. (107.) **OR PRIZE QUESTION.**

By Professor Adrain, Rutger's College, N. Bruns. N. Y.

It is required to determine the time of the very small oscillations of an extremely slender, uniform, and inflexible rectilineal bar, placed horizontally at rest on the surface of an uniform sphere; supposing the point of contact to be exceedingly near the middle of the bar, and the gravitation of the bar towards the sphere to be according to the law of nature.

THE

MATHEMATICAL DIARY,

NO. VII.

BEING THE PRIZE NUMBER OF ROBERT ADRAIN, LL.D. Professor of Mathematics and Natural Philosophy, Rutger's College, New-Brunswick, New-Jersey, HENRY J. ANDERSON, M.D., Professor of Mathematics and Physical Astronomy, Columbia College, New-York, NATHANIEL BOWDITCH, L.L.D., Boston, and EUGENE NULTY, Philadelphia.

DR. ADRAIN'S *Analytical Solution to the Prize Question in No. 5, continued from the last Number.*

Now, if $\frac{ar-f^2}{r}$ be positive let it be denoted by k, and we have

$$z=(a-r)^2+2kv+\frac{f^2v^2}{r^2},$$

of which all the terms are necessarily positive, and therefore z is manifestly a min. when $v=o$, or $y=r$, that is, when the tangent CD is parallel to cr, also the value of e increases continually with v. If $k=ar-f^2=o$, the same consequence follows.

If f^2 be greater than ar, put $\frac{f^2-ar}{r}=k$, and we have

$$z=(a-r)^2-2kv+\frac{f^2v^2}{r^2},$$

in which the term containing the first power of v is necessarily negative; and, consequently, when $v=o$ or $y=r$, the value of z is a maximum.

To examine the general value of z in this case we have

$$\frac{dz}{dv}=-2k+\frac{2f^2v}{r^2},$$

which, equated to zero, gives $v=\frac{kr^2}{f^2}$ or $\frac{y}{r}=\frac{ar}{f^2}$, which,

because ar is less than f^2, gives a point B different from F. In this case $\frac{ddz}{dv^2}=\frac{2f^2}{r^2}$, which shows that the minimum obtains at B.

If now we can seek the value of z, when the point of contact G, and the straight line cr, are on opposite sides of the centre, we obtain nearly as before,

$$z=a^2-b^2+2ay+\frac{f^2y^3}{r^2},$$

from which it is evident, because $2ay$ is necessarily positive, that the greatest value of z is $(a+r)^2$, and the tangent is at P parallel to cr.

Thus, when the tangent cannot meet the given line cr, there may be two maxima, one at F and the other at P, and two minima at B and B′, which four take place when $ar-f^2$ is not negative: but if $ar-f^2$ be negative, the points B and B′ coincide with F, and there is simply a maximum at P and minimum at F.

ANOTHER SOLUTION.—*By the same.*

Let φ= the angle BAF, then the two perpendiculars cc, Dd, are $a-r\cos.\varphi+b\sin.\phi$, and $a-r\cos.\varphi-b\sin.\varphi$; consequently, $z=(a-r\cos.\varphi+b\sin.\varphi)\times(a-r\cos.\phi-b\sin.\varphi)$ that is, $z=(a-r\cos.\varphi)^2-b^2\sin.^2\varphi=a^2-b^2-2ar\cos.\varphi+f^2\cos.^2\varphi$, we have therefore

$$\frac{dz}{d\varphi}=\sin.\varphi(2ar-2f^2\cos.\varphi)=o,$$

whence $\sin.\varphi=o$, or $\cos.\phi=\frac{ar}{f^2}$.

To ascertain which is the maximum and which the minimum, we have

$$\frac{ddz}{d\varphi^2}=\cos.\varphi(2ar-2f^2\cos.\varphi)+2f^2\sin.^2\varphi.$$

Now, when $\sin.\varphi=o$, we have $\frac{ddz}{d\varphi^2}=2(ar-f^2)$, and therefore $\sin.\varphi=o$ gives a maximum or minimum, according as $ar-f^2$ is negative or positive.

Again, if we use $\cos.\phi=\frac{ar}{f^2}$, we have

$\frac{ddz}{d\varphi^2}=2f^2\sin.^2\phi$, which is essentially positive, and shows that the rectangle is a minimum when $\cos.\phi=\frac{ar}{f^2}$.

ARTICLE XIV.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XIII. No. VI.

QUESTION I. (87.)—*By Mr. D. T. Disney, Cincinnati.*

Given $x^2+y^2-x-y=249740$

$xy+x+y=8516$,

to determine x and y without substituting any other values for them.

FIRST SOLUTION.—*By Mr. William S. Denny, Wilmington, Delaware.*

To equation 1st. add twice the 2d, and we have

$(x+y)^2+(x+y)=266772$,

which quadratic, when solved, gives $x+y=516$; by subtracting this from the 2d equation, we have $xy=8000$. Now, having the sum and product of x and y, we find their separate values, $x=500$, $y=16$.

The solutions of Messrs. Devoor V. Burger, Robert Parry, Thomas J. Megear, John Delafield Jun. C. O. Pascalis, and Enoch Lanning.

SECOND SOLUTION.—*By Mr. Enoch Lanning, Morrisville, Bucks Co. Penn.*

By adding the given equations together and adding xy to both sides and extracting the square root, we get

$x+y=\sqrt{(258256+xy)}$,

but from the 2d equation

$x+y=8516-xy$;

by putting these two values of $x+y$ equal to one another,

clearing the equation of radicals, and transposing terms, we get

$$x^2y^2 - 17033xy = -7226100;$$

$\therefore$ by completing the square and extracting the square root, we have

$$xy = 8516.5 \pm 516.5,$$

and by conditions of the second equation 516.5 must be taken as minus; therefore $xy = 8000$, or $x = \frac{8000}{y}$; and by the 2d equation $x = 516 - y$. Put these two values = one another, complete the square and extract the root, and transposition, we have $y = 500$; and $\therefore x = 516 - y = 516 - 500 = 16$.

THIRD SOLUTION.—*Mr. Michael Floy, New-York.*

Put $a = 24974$, and $b = 8516$; then by adding twice the 2d equation to the first, we have,

$$x^2 + 2xy + y^2 + x + y = a + 2b,$$

$$\text{or } (x+y)^2 + (x+y) = a + 2b;$$

$$\text{whence } x + y = -\tfrac{1}{2} \pm \sqrt{(a + 2b + \tfrac{1}{4})} = p(216).$$

Now, from the 2d equation, $xy = b - (x+y) = b - p$; subtracting 4 times this equation from the square of the last, $(x+y=p)$, gives us,

$$x^2 - 2xy + y^2 = p^2 - 4b + 4p;$$

$$\therefore x - y = \pm \sqrt{(p^2 - 4b + 4p)}:$$

Whence, by addition, subtraction, &c.

$$x = \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{(p^2 - 4b + 4p)} = 500,$$

$$\text{and } y = \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{(p^2 - 4b + 4p)} = 16.$$

This question, with the exception of the absolute terms, or right hand members of the equations, is exactly the same as Question 16, page 298, 2d edition, Ryan's Treatise on Algebra.

FOURTH SOLUTION.—*By Mr. George Alsop, Burlington, New-Jersey.*

By adding the equations together, we get $x^2 + xy + y^2 = 258256$, and by adding xy to each side, we get $x^2 + 2xy + y^2 = 258256 + xy$; $\therefore$ by extracting the square root, $x + y = \sqrt{(258256 + xy)}$, and by substituting this value in the 2d equation, we shall find

$$\sqrt{(258256 + xy)} + xy = 8516;$$

whence, by clearing of radicals, squaring, &c. we shall find

$xy=8000$; and $\therefore x+y=516$; hence x and y may be readily found to be 500 and 16, respectively.

QUESTION II. (88.)—*By Mr. Gerardus B. Docharty, Flushing, Long-Island.*

Given $x^4+y^4+x^2+y^2+2x^2y^2=238632$
$y^4+z^4+y^2+z^2+2y^2z^2=1640$
$x^2+y^2+z^2=\sqrt{(275100-x^2-y^2-z^2)}$,

to find the values of x, y, and z.

FIRST SOLUTION.—*By Mr. Robert Parry, Mullica, New-Jersey.*

The first and second equations may be expressed thus:

$$(x^2+y^2)^2+(x^2+y^2)=238632,$$
$$(y^2+z^2)^2+(y^2+z^2)=1640;$$

whence we obtain,

$$x^2+y^2=488, \text{ and } y^2+z^2=40.$$

Now, take the equation, $y^2+z^2-40=0$ from the third given equation, and we get

$$x^2+40=\sqrt{(275100-x^2-y^2-z^2)},$$
$$\text{or } x^4+80x^2+1600=275100-x^2-y^2-z^2$$
$$\text{but } \qquad 40= \qquad\qquad y^2+z^2$$

$\therefore$ by addition $x^4+80x^2+1640=275100-x^2$; hence $x^2=484$, and $x=22$, whence we find $y=2$ and $z=6$.

SECOND SOLUTION.—*By Mr. Devoor V. Burger, Musquito Cove, Long Island.*

The 1st. equation $=(x^2+y^2)^2+(x^2+y^2)=238632$; also, the 2d $=(x^2+y^2+z^2)^2+(x^2+y^2+z^2)=275100$: whence, by completing squares and quadratics, the 1st equation gives $x^2+y^2=488$, or -489; the 2d, $y^2+z^2=40$, or -41; and the 3d, $x^2+y^2+z^2=524$, or -525.

Subtracting the 1st of these new equations from the 3d gives $z^2=36$, or $z=6=$ one No.; also the 2d from the 3d gives $x^2=484$, or $x=22=$ another No.; consequently $y^2=4$, or $y=2$.

THIRD SOLUTION.—*By Mr. John Delafield, Jun. New-York.*

The first equation may be put under the form

$$(x^2+y^2)^2+(x^2+y^2)=238632;$$

completing the square

$$(x^2+y^2)^2+(x^2+y^2)+\tfrac{1}{4}=238632\tfrac{1}{4}\ ;$$

$\therefore$ extracting the root,

$$x^2+y^2+\tfrac{1}{2}=\pm\sqrt{\left(\frac{954529}{4}\right)}=\pm 488\tfrac{1}{2}\ ;$$

$$\therefore x^2+y^2=488\tfrac{1}{2}-\tfrac{1}{2}=488.$$

From the 3d equation, we find

$$(x^2+y^2+z^2)^2+(x^2+y^2+z^2)=275100\ ;$$

$\therefore$ completing the square, extracting the root, &c. we have

$$x^2+y^2+z^2=524\ ;$$

$$\text{but}\quad x^2+y^2\quad\ =488\ ;$$

$\therefore$ by subtraction $z^2=36$ or $z=6$; substituting this value of z in the 2d equation, we have

$$y^4+1296+y^2+36+72y^2=1640\ ;$$

$\therefore$ by transposition, collecting terms, completing the square, &c. we find

$$y^2=4\ ;\ \therefore y=2.$$

But, $x^2+y^2=488$; $\therefore x^2=488-y^2=488-4$, or $x^2=484$; $\therefore x=22$.

Question III. (89.)—*By Mr. Michael Floy, New-York.*

Given the five following equations :

$$x+y+z+v+u=12.15$$
$$x+y+z+vu=\ 9.15$$
$$x+y+zvu=\ 5.68$$
$$x+yzvu=\ 5.09$$
$$xyzvu=\ \ .45,$$

to determine the values of x, y, z, v and u.

Fisrt Solution.—*By Mr. Thomas J. Megear, Wilmington, Delaware.*

After multiplying the 4th equation by x and subtracting the 5th from the resulting equation, we shall have

$$x^2=5.09x-.45\ ;$$

Hence $x=5$.

Again, by substituting this value of x in the 3d and 4th equations, we have, by multiplying the 3d by y and subtracting the 4th,

$$y^2=.68y-.09\ ;\ \therefore y=.5;$$

by proceeding in a similar manner, we find

$$z=.05,\ v=6,\ \text{and}\ u=.6.$$

This question was solved in a similar manner by Messrs. William Vogdes, Enoch Lanning, Robert Parry, William S. Denny, John Delafield Jun., and George Alsop.

SECOND SOLUTION.—*By Mr. D. G. Disney, Cincinnati.*

Subtracting the second equation from the first, the third from the second, the fourth from the third, and the fifth from the fourth, gives

$$v+u-vu=3 \text{ or } u=\frac{3-v}{1-v},$$

$$z+vu-zvu=3.47 \text{ or } uv=\frac{3.47-z}{1-z},$$

$$y+zvu-yzvu=.59 \text{ or } zuv=\frac{.59-y}{1-y},$$

$$x+yzvu-xyzvu=4\ 63 \text{ or } yzvu=\frac{4.63-x}{1-x},$$

$$\text{and } xyzvu=.46\ ;$$

now, it is evident that if we multiply the fourth of these equations by x, the left hand member of the resulting equation is equal to that of the fifth; and if the third equation be multiplied by y, the left hand member of the result shall be equal to that of the fourth, &c. Consequently,

$$\frac{4.63-x}{1-x}.x=.46,$$

$$\text{or } 4.63x-x^2=.46-.46x\ ;$$

$\therefore$ by transposition, completing the squares, &c. we find $x=4.998$ nearly, which being known, the rest may easily be determined.

THIRD SOLUTION.—*By Mr. C. O. Pascalis, New-York.*

Find the values of x in the fourth and fifth equations which will be $x=\frac{.45}{vuzy}$, $x=5.09-vuzy$; putting these values=to one another we have $(vuzy)^2+5.09\ vuyz=-.45$; whence we have $vuzy=.09$: but $x+vuyz=5.09$. $\therefore x=5.09-.09=5$, by repeating this operation and substituting for known quantities their values, we have

$$x=5,\ y=.5,\ z=.05,\ v=6,\ u=.6.$$

QUESTION IV. (90.)—*By Mr. E. Giddings, Fort Niagara.*

The sum of five numbers in geometrical progression, and their fourth difference being given, to determine the numbers by a *quadratic* equation.

FIRST SOLUTION.—*By the Proposer.*

This question is not worded as I at first proposed it, but the mistake was inadvertently made by myself in copying it; it was meant to read thus: *Given the sum of five numbers in geometrical progression* $=62$, *and their fourth difference* $=2$, *to determine the numbers by quadratics.*

Let x, xy, xy^2, xy^3, xy^4, represent the required numbers, then their fourth difference will be

$$x-4xy+6xy^2-4xy^3+xy^4\ ;$$

whence, per question,

$$(y^4+y^3+y^2+y+1)x=62,$$

and $$(y^4-4y^3+6y^2-4y+1)x=2\ ;$$

from these two equations we have by proportion

$$y^4+y^3+y^2+y+1 : y^4-4y^3+6y^2-4y+1 :: 31 : 1\ ;$$

multiplying extremes and means

$$y^4+y^3+y^2+y+1=31y^4-124y^3+186y^2-124y+31\ ;$$

this, by transposition and division, becomes

$$y^4-\frac{125}{30}y^3+\frac{185}{30}y^2-\frac{125}{30}y+1=0\ ;$$

which is a reciprocal equation of the fourth degree, and solved by quadratics agreeably to the rule page 171, Vol. 2, of Bonnycastle's Algebra, $y=2$ or $\frac{1}{2}$, and $x=2$ or 32, and the required numbers are therefore 2, 4, 8, 16 and 32.

SECOND SOLUTION.—*By Mr. James Foster, late of Belfast, Ireland.*

Let $x=$ the first term $r=$ the ratio, $s=$ the sum of the terms, and $d=$ the fourth difference. Then, $x+rx+r^2x+r^3x+r^4x=s$, $x-4rx+6r^2x-4r^3x+r^4x=d$. By subtracting the second equation from the first and dividing the result by 5, we have $rx-r^2x+r^3x=\frac{s-d}{5}$. Again, by subtracting this last from the first gives $x+2r^2x+r^4=\frac{4s+d}{5}$; therefore $x=\frac{4s+d}{5(r^4+2r^2+1)}$, and from the second

equation x is found $=\frac{d}{r^4-4r^3+6r^2-4r+1}$. Whence $\frac{4s+d}{5(r^2+1)^2}=\frac{d}{(r-1)^4}$; $\therefore$ extracting the square root, &c. $\surd\left(\frac{4s+d}{5d}\right)=\frac{r^2+1}{(r-1)^2}$; therefore, putting $\surd\left(\frac{4s+d}{5d}\right)=a$, and clearing of fractions, $ar^2-2ar+a=r^2+1$;

by transposition, $(a-1)r^2-2ar=1-a$;

by division, $r^2-\frac{2a}{a-1}r=\frac{1-a}{a-1}=-1$;

$\therefore$ completing the square, extracting the root, &c.

$$r=\frac{a}{a-1}\pm\surd\left(\frac{2a-1}{(a-1)^2}\right),$$

$$\text{or } r=\frac{a\pm1}{a-1}\surd(2a-1).$$

This question was solved exactly in the same manner, by Messrs. John Delafield Jun. C. O. Pascalis, Farrell Ward, Mary Bond, William J. Lewis, and George Alsop.

THIRD SOLUTION.—*By Mr. Gerardus B. Docharty, New-York.*

Let $\frac{x}{r^2}$, $\frac{x}{r}$, x, xr, xr^2, be the five Nos. in geom. prog. $m=$ their sum, $d=$ their fourth difference;

then $\frac{x}{r^2}+\frac{x}{r}+x+xr+xr^2=m$

and $\frac{x}{r^2}-\frac{4x}{r}+6x-4xr+xr^2=d$

$\frac{5x}{r}-5x+5xr=m-d$ by subtraction,

or $\frac{x}{r}-x+xr=\frac{m-d}{5}=c$; $\therefore\frac{1}{r}-1+r=\frac{c}{x}$, $r+\frac{1}{r}=\frac{c}{x}+1$.

But, by second equation, $\frac{1}{r^2}-\frac{4}{r}+6-4r+r^2=\frac{d}{x}$, or, $r^2-4r+6-\frac{4}{r}+\frac{1}{r^2}=\frac{d}{x}$; hence by extracting the root on both sides $r-2+\frac{1}{r}=\surd\frac{d}{x}$, or, $r+\frac{1}{r}=\surd\frac{d}{x}+2$, these two values

of $r+\frac{1}{r}$ put = give $\sqrt{\frac{d}{x}}+2=\frac{c}{x}+1 \therefore \sqrt{\frac{d}{x}}=\frac{c}{x}-1 \therefore \frac{d}{x}=\frac{c^2}{x^2}$ $-\frac{2c}{x}+1$ and $xd=c^2-2cx+x^2 \therefore x^2-(2c-d)x=-c^2$, put $-(2c-d)=-2b$, then $x^2-2bx+b^2=b^2-c^2$, then $x-b=\sqrt{(b^2-c^2)}$; hence $x=b\pm\sqrt{(b^2-c^2)}$.

Fouth Solution.—*By Mr. Henry Darnall, Philadelphia.*

We have $\frac{1+x+x^2+x^3+x^4}{1-4x+6x^2-4x^3+x^4}=a$ or $\frac{\frac{1}{x^2}+\frac{1}{x}+1+x+x^2}{\frac{1}{x^2}-\frac{1}{x}+6-4x+x^2}$ $=a$, let $x+\frac{1}{x}=y$, $x^2+2+\frac{1}{x^2}=y^2$, $x^2+1+\frac{1}{x}=y^2-1$, $x^2+6+\frac{1}{x}=y^2+4$ and $x+1=xy$, hence $\frac{y^2+y+1}{y^2-4y+4}=a$, a quadratic from which y will be known and then x.

The solutions of Messrs. James Hamilton and Michael Floy, were exactly like this.

Fifth Solution.—*By Mr. John Swinburn, Brooklyn.*

Let x denote the first term and y the ratio: also, let s denote their sum and d their fourth difference; then per question, we have

$$x+xy+xy^2+xy^3+xy^4=s$$
$$\text{and } xy^4-4xy^3+6xy^2-4xy+x=d$$

Subtract the second equation from the first, and we will have $5xy^3-5xy^2+5xy=s-d$; or, by dividing by $5xy$,

$$y^2-y+1=\frac{s-d}{5xy}$$

The second equation, divided by x and square root extracted gives $y^2-2y+1=\frac{\sqrt{d}}{\sqrt{x}}$;

$\therefore$ by subtraction $y=\frac{s-d}{5xy}-\frac{\sqrt{d}}{\sqrt{x}}$,

and, by an obvious reduction $x^2y^4-xy^2\left(\frac{2s+3d}{5}\right)=-\left(\frac{s-d}{5}\right)^2$; from which, by quadratics, xy^2 can be obtained,

which put $=a$; then $x=\frac{a}{y^2}$; this substituted for x in the equation $y^2-y+1=\frac{s-d}{5xy}$, and reduced, gives $y^2-(1+\frac{s-d}{5a}y=-1$, to find y and hence the rest.

QUESTION V. (91.)—*By Mr. Selah Hammond.*

The perpendicular from the right angle of a triangle upon the hypothenuse is equal to the product of the legs, and makes the greater to the lesser segment as 3 to 2; to determine the triangle.

FIRST SOLUTION.—*By Mr. William Vogdes, Edgemont, Delaware Co. Penn.*

Let $x=$ the base, $y=$ the perpendicular, and $z=$ hypothenuse; then, by the question, $xy=$ the perpendicular falling upon the hypothenuse from the right angle.

Now, by similar triangles, we have

$z : x :: y : xy$; $z=1=$ hypothenuse;

also, $1 : y :: y : y^2=$ the greater segment,

and $1 : x :: x : x^2=$ the less;

$\therefore x^2+y^2=1$, by the question;

and $y^2 : x^2 :: 3 : 2$;

hence $y^2=\frac{3x^2}{2}$;

Substituting this value of y^2 in the above equation, we have

$$x^2+\frac{3x^2}{2}=1\ ;\ \therefore x=\sqrt{\frac{2}{5}}=\text{one leg};$$

and consequently $y=\sqrt{\left(\frac{3x^2}{2}\right)}=\sqrt{\frac{3}{5}}=$ the other.

SECOND SOLUTION.—*By Mr. Alpheus Bixby, New-York.*

Let z and v represent the segments of the base, x and y the legs, and a the area of the triangle; then $xy=2a$, and by the first condition of the question, $xy=p$, the perpendicular; therefore, $p=2a$; but $pz+pv=2a$; $\therefore$ by substitution,

$$2az+2av=2a,$$

and by division, $z+v=1=$ the hypothenuse. By the se-

cond condition of the question, $z : v :: 3 : 2$; $\therefore 2z = 3v$; but from the last equation, $v = 1 - z$; therefore, by substitution,

$$2z = 3(1-z) = 3 - 3z : \text{ consequently,}$$

$$z = \frac{3}{5}, \text{ and } v = 1 - z = 1 - \frac{3}{5} = \frac{2}{5}.$$

Again, by similar triangles,

$$1 : x :: x : \frac{3}{5}; \ \therefore x^2 = \frac{3}{5}, \text{ and } x = \sqrt{\frac{3}{5}};$$

$$1 : y :: y : \frac{2}{5}; \ \therefore y^2 = \frac{2}{5}, \text{ and } y = \sqrt{\frac{2}{5}}.$$

THIRD SOLUTION.—*By Mr. William Kells, Bergen.*

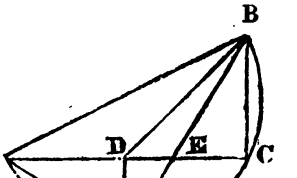
Let $3x$ and $2x$ = the segments of the hypothenuse, and y = the perpendicular; then $6x^2 = y^2$, by the question. Again, $9x^2 + y^2$ = the square of one leg, and $\therefore \frac{y^2}{9x^2+y^2}$ = the square of the other; hence $\frac{y^2}{9x^2+y^2} + 9x^2 + y^2 = 25x^2$; by substituting the above value of y^2 in this equation, we shall find, after proper reduction, $x = \frac{1}{5}$; $\therefore y^2$ is readily found to be equal to $\frac{6}{25}$. Consequently, the legs are also given $= \sqrt{\frac{2}{5}}$ and $\sqrt{\frac{3}{5}}$.

QUESTION VI. (92.)—*By Mr. S. Wright, Bucks Co. Penn.*

In a right-angled triangle, there is given the difference between the lines bisecting the base and vertical angle, and the part of the base between those two lines, to determine the triangle.

FIRST SOLUTION.—*By Mr. Eugene Nulty, Phil.*

Let ABC be the triangle, BD the line bisecting the base, and BE the line bisecting the vertical angle. On AB as a diameter describe the semicircle ABC; draw DF perpendicular to AC meeting the arc AC and the prolongation of BE in F. Join AF, and put A equal to the given difference of BD,



BE, b=DE, ρ=BE, and ϕ= the angle BEC. In the triangle BDE we have $BD^2-BE^2=CD^2-CE^2$, or $(\rho+a)^2-\rho^2=(\rho\cos.\phi+b)^2-\rho^2\cos.\varphi^2$, which reduced gives $\rho=\frac{a^2-b^2}{2(b\cos.\varphi-a)}$, the equation of one hyperbola, and the locus of the vertex B.

In the triangles BCE, DEF, which are similar, we have $BC^2:CE^2::DF^2=AD,DE:DE^2::AD:DE$, or $\sin.\varphi^2:\cos.\varphi^2::\rho\cos.\phi+b:b$, from which by composition and reduction, we get $\rho=\frac{b(1-2\cos.\varphi^2)}{\cos.\varphi^3}$. Equating this value of ρ with that before found, we have

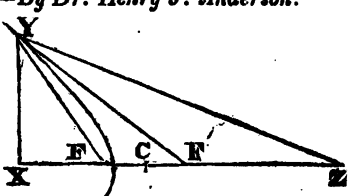
$$(a^2+3b^2)\cos.\varphi^3-4ab\cos.\varphi^2-2b^2\cos.\phi+2ab=0,$$

which, resolved, gives $\cos.\varphi=m$, and consequently, $\rho=\frac{a^2-b^2}{2(bm-a)}=m'$, $EC=mm'$, by virtue of which values the vertex B may be determined, and the triangle ABC.

SECOND SOLUTION.—*By Dr. Henry J. Anderson.*

Let $2a$ and $2c$ be the given quantities respectively.

Imagine an hyperbola with semitransverse = a, and focal distance = FC=c, its equation will be $y^2=\frac{b^2}{a^2}(x^2-a^2)$ where $a^2+b^2=c^2$, CX=x, XY=y, it only remains to find the point Y so that $\angle XYZ=2\angle XYF$, F and F′ being the foci, C the centre, and F′Z=F′X.



By trigonom. 2. cot. 2φ= cot. ϕ − tan.ϕ

Therefore, $\frac{y}{x+c}=\frac{y}{x-c}-\frac{x-c}{y}$

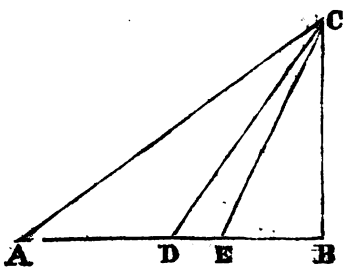
or, $2cy^2=(x-c)(x^2-c^2)$

Therefore, $2c\frac{b^2}{a^2}(x^2-a^2)=(x-c)(x^2-c^2)$

and $x^3+c\frac{2c^2-a^2}{a^2}(x^2+c^2x-3c^3+2a^2c=0$ a cubic.

THIRD SOLUTION.—*By* Ομικρον, *North Carolina.*

In the marginal diagram, let ABC be the required triangle, CD the line bisecting the base, and EC the line bisecting the vertical angle, and let AD$=x$, DC$=y$, ED$=a$, DC$-$EC$=b$; then $2x^2+2y^2=$AC$^2+$BC^2, $x^2-y^2=$BC^2; hence $3x^2+y^2=$AC^2, AE$^2+$EC$^2+2$AE. EB$=(x+a)^2+2(x+a)(x-a)$, whence $y=\frac{2ax+b^2-a^2}{2b}$.



Again, $x+a : x-a :: (3x^2+y^2)^{\frac{1}{2}} : (y^2-x^2)^{\frac{1}{2}}$ or $(x+a)^2 : (x-a)^2 :: 3x^2+y^2 : y^2-x^2$; whence, dividendo et componendo, $ax : x^2-ax+a^2 :: x^2 : y^2$, consequently $x^2(x^2-ax+a^2)=ax.y^2=ax\left(\frac{2ax+b^2-a^2}{2b}\right)^2$ or, $x^3-ax^2+a^2x=a\left(\frac{2ax+b^2-a^2}{2b}\right)^2$ a cubic equation from which x may be found and thence y, and the several sides of the triangle.

FOURTH SOLUTION.—*By Mary Bond, Frederick, Maryland.*

Let ABC (see the diagram to the last solution) be the right angled triangle required, CD the line drawn from the vertical angle bisecting the base in D, and CE the line bisecting the vertical angle. By the question DE$=b$, and CD$-$CE$=2a$. Let now CD$=x+a$, and CE$=x-a$, and since CD$^2=(a+x)^2=$DE$^2+$EC$^2+2$DE$\times$EB, we have EB$=\frac{4ax-b^2}{2b}$, and thence AD$=$DB$=\frac{4ax+b^2}{4b}$, and AE$=\frac{4ax+3b^2}{2b}$.

Again AC^2 : CB^2 :: AE^2 : EB^2 or AC$^2-$BC$^2=$AB^2 : BC^2 :: AE$^2-$EB$^2=$AB$\times$2DE : EB^2; that is, AB$=\frac{4ax+b^2}{b}$: BC$^2=(x-a)^2$

$-\left(\frac{4ax-b^2}{2b}\right)^2 :: 2\text{DE}=2b : \text{EB}^2=\left(\frac{4ax-b^2}{2b}\right)^2$ Whence by reduction we shall find

$$x^3+\frac{(2a^2-b^2)\,b^2}{8a^3}.x^2-\frac{b^4}{16a^2}.x+\frac{(3b^2-3a^2).b^4}{64a^3}=0,$$

from which cubic equation x may be found.

The solutions of Messrs. James Macully and James Foster, were similar to this.

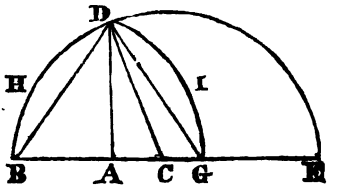
QUESTION VII. (93.)—*By Mr. James Hamilton, Trenton.*

What is the radius of a circle whose centre being taken in the circumference of another containing an acre, shall cut off half its contents?

FIRST SOLUTION.—*By Professor Strong, Hamilton College.*

From the conditions of the question I have sin. $\phi-\varphi$ cos. $\phi=\rho$ ($\rho=$ the length of the quadrant rad. (1) and $\varphi=$ the angle made by the radii of the circle sought drawn to the points of intersection of the two circles). I find that $\varphi=109°\ 11'\ 18''$, and the radius of the circle sought equal to 8.2692 rods nearly.

SECOND SOLUTION.—*By Nathaniel Bowditch, L. L. D. Boston.*



Let CB$=a$ be the radius of the proposed circle whose area is an acre, B the centre, and DG the arc of the required circle whose radius is r, such that the area BHDIG$=\frac{1}{2}$ semicircle BDE.

Put angle BCD$=2x$, angle GBD$=90-x$, and draw the sine DA common to both arcs, we get

$$r=2a \text{ sin. } x.$$

But the area of the segment BHDA$=a^2(x-\frac{1}{2}$ sin. $2x$. cos. $2x)$ and that of ADIG$=r^2(45-\frac{1}{2}x-\frac{1}{2}$ cos. x. sin. $x)$ $=2a^2$. sin. r^2 $(45°-\frac{1}{2}x-\frac{1}{2}$ sin. $2x)$. Adding these equations gives the area BHDIG$=a^2(180°-x-$ sin. $2x-\frac{1}{2}$sin. $2x$. cos. $2x)$. The double of which, or a^2 $(2\pi-2x-2$

sin. $2x$ − sin. $2x$ cos. $2x$) = area semicircle BDE = $\frac{1}{2}a^2\pi$ (π being = 3. 14159 −) dividing by a^2 we get $2x + 2$ sin. $2x$ + sin. $2x$. cos. $2x = \frac{3}{2}\pi$, whence by trials we easily find x, and then $r = 2a$ sin. x.

THIRD SOLUTION.—*By Robert Adrain, LL. D.*

In a circle to radius unity let $2z$ be the arc of which the chord is the required radius, then π being the area of the given circle to radius unity, if we express analytically the area cut off by the radius sought and divide by 2, we obtain the transcendental equation

$$\left(\frac{\pi}{2} - z\right) \text{cos. } 2z + \tfrac{1}{2} \text{ sin. } 2z = \frac{\pi}{4},$$

Hence $z = 35° 24'$, and therefore if R = the radius of the given circle, the radius sought = 2R sin. $(35° 24') = 1.158$R.

FOURTH SOLUTION.—*By Dr. Henry J. Anderson.*

Let the radius of the given circle be represented by unity, and of the two portions of its circumference terminated at the intersections of the two circles let the greater be denoted by 2φ. Then, by the rules of mensuration, it will be found that two equal parts into which the given circle is divided are equal, each to 2φ cos.$^2\frac{1}{2}\varphi + \pi - \varphi -$ sin. φ. Putting this equal to $\frac{1}{2}\pi$, the area of the semicircle, and transposing we have

$$\text{sin. } \varphi - (2 \text{ cos.}^2 \tfrac{1}{2}\varphi - 1) = \frac{\pi}{2}, \text{ or by trigonom.}$$

$$\text{sin. } \phi - \varphi \text{ cos. } \phi = \frac{\pi}{2};$$

whence $\varphi = 109° 11' 17''$ and the required radius = 1.15874.

If the content of the given circle be one acre, then the required radius will be 206.7336 links, or about 45.4814 yards.

QUESTION VIII. (94.)—*By Mr. D. W. Carmody, N. Y.*

It is required to find an arc of a circle, such, that if a perpendicular be let fall from the extremity of the sine on the tangent, and the secant to the point where the perpendicular meets the tangent be drawn, the rectangle under the versed sine and part of the secant between the centre and the sine is to the rectangle under the cosine

and that part of the sine between the arc and secant in a given ratio as $m : n$. And show the geometrical analysis with the construction.

First Solution.—*Mr. B. Mc. Gowan, N. Y.*

BG. AL : AG . CL :: m : n ;

Analysis.

(CE) BG : CL :: ED (BA or R) : DA (EB),

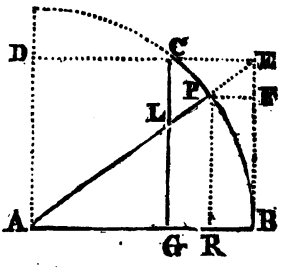
AL : AG :: AE : AB (R),

AL.BG : AG.CL :: (ED) R.

AE : EB.R :: AE : EB ;

∵ AE : EB :: m : n ;

but AE : EB :: R : PR or sine.



Composition.

Take AB : BF :: m : n, draw FP parallel to AB, join AP and produce it to meet the tangent BE in E, draw ED and CG parallel to AB and AD ; BC is the arc required.

Cor. 1. Taking $m=n$, the tangent becomes parallel to the sine and the arc becomes a quadrant or 90° ; ∵ the points GR and A coincide, the cosine becomes 0, and versed sine =radius.

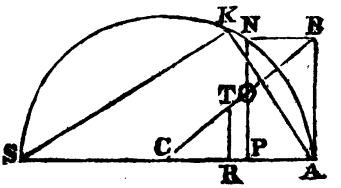
Cor. 2. Taking m less than r, rad. will be less than sine ; hence, m must be taken less than n in all cases.

Second Solution.—*By Academicus, N. Y.*

By similar triangles,

NB : CP :: NO : OP & ∴ $\frac{NB}{CP}=\frac{NO}{OP}$, but by the question. $\frac{NB}{CP}\times\frac{CO}{NO}=\frac{m}{n}$

∴ $\frac{NO}{OP}\times\frac{CO}{NO}$ or $\frac{CO}{OP}=\frac{m}{n}$,



and by similar triangles $\frac{CO}{OP}=\frac{CB}{AC}=\frac{m}{n}$, it thence appears that we have the base AC of a right angled triangle and the ratio of the hyp. and the other side, to construct it. This is easily effected by making $m : n$ :: AB : SK, then

the triangle SKA is manifestly similar to the required one; then say, as SK : KA :: AC : AB, join CB, and ABC is the triangle required of which the side AB=NP is the arch required: draw BN parallel to AC, which determines the arch AN.

THIRD SOLUTION.—*By Mr. Charles Potts, Philadelphia.*

Suppose BC, (see the fig. to the 1st solution) to be the arc required, and the figure being completed we have by the question

$$BG \times AL \text{ or } CE \times AL : AG \times LC :: m : n,$$

and because of the similar triangles ALG, ELC, CE. LG=CL. AG, multiply both sides of the equation by AL and turning it into an analogy, we shall have, CE.AL : AG. CL : : AL : LG : : m : n. Hence if we call the radius unity and the sine of the arc BC=x, then $AE=\frac{mx}{n}$, and (Euc. 47. 1) $x^2 + 1 = \frac{m^2 x}{n^2}$; $\therefore x = \frac{n}{\sqrt{(m^2 - n^2)}}$.

FOURTH SOLUTION.—*By Mr. James Macully, N. Y.*

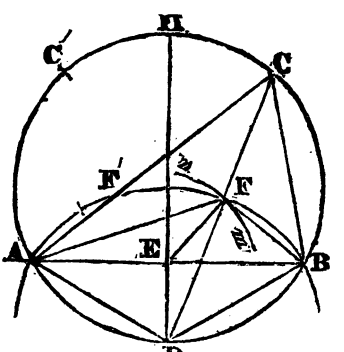
Let ANS be the given circle, (see the figure to the 2d solution) it is required to find an arc AN answering the conditions of the question.

Analysis. Suppose the thing done, and that CB is the secant joining the centre and the point B on the tangent where the perpendicular from the extremity of the sine meets it, and NP the sine. Now the triangles COP and NOB are similar; hence we have PC : CO : : AC : CB and ON : NB : : BA : AC; multiplying these proportions and rejecting AC, we have PC $\times$ ON : CO $\times$ NB : : BA : BC. But by the question we have PC $\times$ ON : CO $\times$ NB : : n : m; whence BC : BE : : m : n. Hence this construction, make on the given radius CA, CR=$\sqrt{(m^2 - n^2)}$ and perpendicular to it RT=n, join CT, and produce it to meet the tangent in B, from B draw the perpendicular BN cutting the circumference in N; then AN is the required arc. The demonstration is sufficiently obvious from the analysis.

QUESTION IX. (95.)—*By Mr. James Phillips.*

Given the base, the vertical angle, and the line drawn from the centre of the inscribed circle, bisecting the base, to determine the triangle.

FIRST SOLUTION.—*By Mr. B. Mc. Gowan, New-York.*



Describe a segment of a circle on the given base AB, containing an angle = the supplement of the given angle. Bisect the base AB, and erect the perpendicular ED, join DB, from the centre D, DB as radius, describe an arc of a circle, also from the centre E, the given line drawn from the centre of the inscribed circle to the middle of the base, as radius describe another arc, cutting the former in F and F', join DF and produce the line DF to cut the circle in C, join AC and BC, ABC will be the triangle required.

Because the angle ACB= the supplement of the angle ADB= the given vertical angle. The lines DB, DF, and DA are equal, F or F' is the centre of the inscribed circle, (Bland's Problems;) and EF= the line given; hence, ACB or AC'B is the required triangle.

SECOND SOLUTION.—*By Mary Bond, Fred. Md.*

On the base AB (see the last Fig.) describe a segment of a circle ACB to contain an angle equal to the one given. From E the middle of AB draw ED perpendicular to AB, and from D as a centre and distance DB describe the arch AFB. Again, from E as a centre and distance equal to the given distance from the centre of the inscribed circle to the middle of the base, describe an arch cutting the arch AFB in the point F. Join DF which produce to C, and joining AC, CB, ACB shall be the triangle required. It has often been proved that the centre of a circle inscribing any triangle that may be formed in the segment ACB on the base AB will be found in the arch AFB; hence, as EF is equal to the given distance, and because AD=BD, DFC bisects the vertical angle, F is the centre of the circle inscribing the required triangle ACB.—*Calculation.*—In the

triangle FED all the sides are known, consequently the angle EDF is known; whence BDC=BAC is known. In the same manner ADC=ABC becomes known; now in the triangle ABC all the angles and base AB are known; consequently AC and CB become known.

THIRD SOLUTION.—*By Mathetus, Bucks Co. Penn.*

Let AB (see Fig. to solution 1st.) be the given base of the triangle. On it describe a segment of a circle, ACB, containing an angle equal to the given vertical angle. Again, on AB describe a segment AFB that may contain an angle equal to the supplement of ½ the angles at the base; then from the point E, the middle of the base with the given distance EF (the line to be drawn to the centre of the inscribed circle,) describe an arc intersecting AFB in F. Join AF, BF, also draw the lines AC, BC, doubling the angles FAE, FBE; and they will meet in the periphery of the circle ACB: the truth of this construction is too obvious to need a demonstration.

FOURTH SOLUTION.—*By Mr. Edward Giddings.*

Construction.—Upon AB (see Fig. 1st solution) the given base, describe the segment AHB capable of containing the given vertical angle. complete the circle and draw the diameter DH at right angles to AB, which it will bisect in E with E as a centre, and the given distance between the centre of the inscribed circle and the centre of the base, describe the arch *mn*, and on D as a centre with the radius DB describe the arch AFB, intersecting *mn* in F, from D through F draw DC, intersecting the periphery in C, join AC, CB, and ABC will be the triangle required.

Demonstration—Join AD, BD, and FB, then, because the diameter DH is perpendicular to AB, it bisects it in E, and it also bisects the arch ADB in D, and the angle ACB is therefore bisected by CD; again, because D is the centre of the circle AFB, $\angle DFB = \angle DBF$, but, $\angle DFB = \angle DCB + \angle CBF$, and $\angle DCB = \angle DAB = \angle DBA$, $\therefore \angle DBA + \angle ABF = \angle DBA + \angle CBF$, the angle ABC is therefore bisected by the line BF, and F is therefore the centre of the inscribed circle; lastly AB is the given base, FE is the given distance between the centre of the inscribed circle and the middle of the base, and ABC is the given vertical angle, ABC is therefore the triangle required.

Calculation.—In the isosceles triangle ADB are given the base AB and all the angles to find DB=DF, and the perpendicular DE, and in the triangle EFO are given all the sides to find the angle EDF, whence $\angle BDC = \angle BAC$ becomes known; lastly, in the triangle ABC are given all the angles and the side AB to find AC and CB.

QUESTION X. (96.)—*By Joseph C. Strode.*

To find an arc such that its cosine, sine, and tangent, shall be in harmonical proportion.

FIRST SOLUTION.—*By William J. Lewis, Wilmington.*

Let x= cosine of the required arc, radius being unity; then $\sqrt{(1-x^2)}$= sine, and $\frac{\sqrt{(1-x^2)}}{x}$= tangent.

Hence, by the question and the properties of harmonic proportion, $\sqrt{(1-x^2)} = \frac{2\sqrt{(1-x^2)}}{x^2+\sqrt{(1-x^2)}}$; $\therefore 5x^2-4x^3+x^4 = 1$ and $x=.57428$. Hence the arc required is 54° 57′ nearly.

SECOND SOLUTION.—*By Mr. Farrell Ward, New-York.*

Let y= cosine, x= sine, and $y : x :: 1 : \frac{x}{y}$=tangent; then, according to the nature of harmonic proportion, $y : \frac{x}{y} :: y-x : x-\frac{x}{y}$, from this, $xy-x=x-\frac{x^2}{y}$, or $y^2-2y = -x = = \sqrt{(1-y^2)}$, and squaring, we have $y^4-4y^3+5y^2 = 1$, this solved either by approximation, or by the rule of *Des Cartes* for biquadratics, will give the value of y or the cosine of the required arc.

THIRD SOLUTION.—*By Mr. James Hamilton, Trenton.*

Put x= cosine, y= sine and radius =1; $\therefore x^2+y^2 = 1$ and $\frac{y}{x}$ = tangent. By harmonics, $\frac{1}{x}+\frac{x}{y}=\frac{2}{y}$ or $y=2x-x^2$. From the 1st equation, $y=\sqrt{(1-x^2)}$; $\therefore$ comparing these values of y, we have $2x-x^2=\sqrt{(1-x^2)}$ or $x^4-4x^3+5x^2=1$. Hence the value of x may be readily found, and consequently the arc required is also given.

FOURTH SOLUTION.—*By Mr. Eugene Nulty, Phil.*

The reciprocals of quantities in harmonic proportion are in arithmetical progression, wherefore putting $\varphi =$ the required arc, $\frac{1}{\cos .\varphi} + \frac{1}{\tan .\phi} = \frac{2}{\sin .\phi}$ or tan. $\phi +$ cos. $\varphi = 2$. This solved by a table of natural tangents and co sines gives $\phi =$ arc. $(= 54° 57'$ nearly.)

The solutions of Dr. Henry J. Anderson and Dr. Adrain were similar to this.

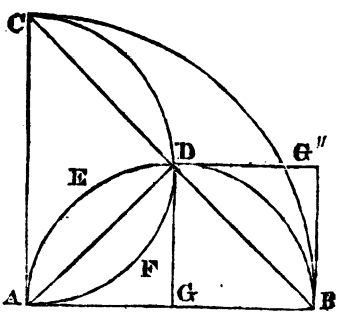
FIFTH SOLUTION.—*By Nemo, N. Y.*

Let $c =$ cosine and $t =$ tangent to radius unity, then (per question) $\frac{2ct}{c+t} = ct$; $\therefore c+t=2$, and by inspection the arc $= 54° 57'$ nearly. The same conclusion is arrived at by eliminating t or c, and reducing the resulting equation in the common way.

QUESTION XI. (97.)—*By Mr. James Divver. S. C.*

If on each of the radii AB, AC, of a quadrant of a circle, semicircles AEDB, AFDC, be respectively described, cutting each other in the points A, D, (the point A being the centre of the circle,) the area included by the two arcs AED, DFA, will always be equal to the area included by the two arcs CD, DB, and the quadrantal arc BC. And the area included by the arcs AFD, DB, and the right line AB, will be equal to the square of the right line drawn from the point D to the middle of AB.

FIRST SOLUTION.—*By Mr. Nathan Brown, Whitingham, Vermont.*



The circle whose radius is AB=four times that whose radius is $\frac{1}{2}$ AB. For circles are to each other as the squares of their radii. Hence the semicircle ADB, ADC are together equal to the quadrant ACB. But ADB+ADC — AEDF + CDB=ACB; wherefore AEDF=CDB.

Also if AD, CD, DB be joined, it will be seen that DG is at right angles to AB. Suppose the figure AFDG to be turned round the point D till the quadrantal arcs AFD, BD coincide; then will DGBG″ be a square. Therefore AFDB $=DG^2$.

This was solved in the same manner exactly by Mr. Charles Potts, Phil. and James Macully, N. Y.

SECOND SOLUTION.—*By Mr. James Sloane, N. Y.*

It is evident from the first elements of geometry that the square of the half of any line is equal to the square of one fourth of the square of the whole and all similar surfaces being to each other as the squares of their like parts, it follows that the semicircle described upon the side of a quadrant must be equal in area to half that quadrant; hence the two semicircles are equal to the quadrant; but the part AEDFA (see the diagram to the first solution) is common to both, and the part BDCB is omitted; therefore BDCB is equal to the common part AEDFA.

And that the area included by AFD, DB, and the right line AB, will be equal to the square of the right line DH is also very evident by inspecting the figure; for the square BG″ DG includes a space BG″D exterior to the given space which is equal to the space AGDH, as is plain from these areas being contained under equal and similar lines similarly posited.

THIRD SOLUTION.—*By Mr. Alpheus Bixby, N. Y.*

Circles are to each other as the squares of their diameters. Put $x=$ the radius AC of the circle whose quadrantal arc is BC, (see fig. to sol 1st) $=$ the diameter of the circle whose half circumference $=$AFDC; then $2x=$ the diameter of the circle whose quadrantal arc is BC; $\therefore x^2 : 4x^2 ::$ the area of the less circle : the area of the greater, or area of the less circle : area of the greater :: 1 : 4; hence the area of the quadrant ABC$=$ areas of the two equal semicircles AEDB, AFDC; $\therefore$ the semicircle AEDB being half the quadrant, the area AFDCBA is also $=$ equal half the quadrant. Take away the common area AFDBA from both these equals and there will remain AEDFA$=$EDCB. Draw the right lines AD, DB; now the area of the segment AFD is evidently equal to the area of the segment on DB, the one being included and the other excluded from

the triangle ADB ; ∴ the area AFDBA= the triangle ADB. Put p= the perpendicular and r= the radius of the less circle ; then $r=p=\frac{1}{2}$ base of the triangle, and consequently $p^2=pr$= area of the triangle ADB=area of the space AFDBA.

QUESTION XII. (98.)—*By Mr. James Macully, N. Y.*

Given the vertical angle of a plane triangle, the straight drawn from its centre of gravity to the base, and the ratio of its greatest inscribed ellipse to that of its greatest inscribed parabola ; to determine the triangle.

FIRST SOLUTION.—*By Mr. Eugene Nulty, Philadelphia.*

This question is indeterminate for the ratio of the area of the greatest inscribed ellipse, being to that of the greatest inscribed parabola as $4\pi : 9$ involves no part of the required triangle. The proposer must therefore have meant the product of these areas, (or some other function of them,) and this being presumed the case, the question may be resolved as follows :

Let ABC be the triangle, (the figure can be readily supplied by the reader,) BD a line drawn to the middle of the base, and therefore passing through the centre of gravity. Draw BE perpendicular to the base ; and put BD $=a=$ three times the given distance between the centre of gravity and the point D ; the product of the areas of greatest inscribed ellipse and parabola $=b^2$, half the base $=x$ and the angle BDE$=\varphi$.

In the triangle ABC, the angles A+B+C$=\pi$, and therefore tan. A+tan. B+tan. C=tan. A. tan. B, tan. C. But tan. $A=\frac{BE}{AE}=\frac{a \sin. \phi}{x+a \cos. \phi}$, tan $C=\frac{a \sin. \varphi}{x-a \cos. \varphi}$ and tan. B$=c$, a given quantity : wherefore by substitution and reduction we get $2ax \sin. \varphi=c(x^2-a^2)$. Again, the areas of the greatest inscribed ellipse and parabola are known to be $\frac{\sqrt{3}}{9} . a\pi x \sin. \phi$ and $\frac{\sqrt{3}}{4} . ax \sin. \varphi$ respectively ; wherefore $\frac{a^2\pi}{12}x^2 \sin. \phi^2=b^2$ an dsin. $\varphi=\sqrt{\frac{12}{\pi}} . \frac{b}{ax}=\frac{m}{x}$, by virtue of which the preceding equation becomes $2am=c(x^2-a^2)$ and consequently $x=\sqrt{\left(\frac{a^2c-2am}{c}\right)}=m'$,

and sin. $\phi = \frac{n}{m} = m''$, by means of which values the triangle may be determined.

Dr. Anderson's solution was somewhat similar, and went on the hypothesis that the author meant the sum or difference of the inscribed figures.

QUESTION XIII. (99.)—*By Mr. Charles Farquhar, Alex. D. C.*

Find values of x and y, so that $(x^n \pm y^n)^{\frac{1}{n+1}}$ shall be rational and integral, n being any positive number whatever.

FIRST SOLUTION.—*By Mr. Charles Potts, Philadelphia.*

Let $x = az$ and $y = bz$, then the given expression $(x^n \pm y^n)^{\frac{1}{n+1}}$ will be transformed into $(a^n z^n \pm b^n z^n)^{\frac{1}{n+1}}$; therefore if we make $(a^n z^n \pm b^n z^n)^{\frac{1}{n+1}} = cz$ and raise both sides to the power denoted by $n+1$, we shall have $a^n z^n \pm b^n z^n = c^{n+1} z^{n+1}$; $\therefore$ if we divide by z^n we have $a^n \pm b^n = c^{n+1} z$ or $z = \frac{a^n \pm b^n}{c^{n+1}}$. Hence $x = \frac{a^{n+1} \pm ab^n}{c^{n+1}}$ and $y = \frac{a^n b \pm b^{n+1}}{c^{n+1}}$; in which a, b, c, may be taken at pleasure.

SECOND SOLUTION.—*By Mr. B. Mc. Gowan, New-York.*

$(x^n \pm y^n)^{\frac{1}{n+1}} = ay$. Assume $by = x$, then $x^n \pm y^n = a^{n+1} y^{n+1}$, or $b^n y^n + y^n = a^{n+1} y^{n+1}$; $\therefore y = \frac{b^n + 1}{n+1}$, and $x = \frac{b(b^n \pm 1)}{a^{n+1}}$, in which a and b may be taken at pleasure.

THIRD SOLUTION.—*By John Delafield, Jun.*

To make $(x^n \pm y^n)^{\frac{1}{n+1}}$ rational.

Put $(x^n \pm y^n)^{\frac{1}{n+1}} = y$. Then, $x^n \pm y^n = y^{n+1}$, and by transposition, $x^n = y^{n+1} \pm y^n$. Now, since x^n is already rational, we have only to make $y^{n+1} \mp y^n$ rational, or, $y \pm 1 = r^n$ or $y = r^n \pm 1$, where r may represent any number at pleasure provided it be greater than unity.

FOURTH SOLUTION.—*By Academicus, N. Y.*

Let sy be written for x in the given expression and it

becomes $(s^n y^n \pm y^n)^{\frac{1}{n+1}}$, which, in order to render rational, equate to py, and then $s^n y^n \pm y^n = p^{n+1} y^{n+1}$; hence $y = \frac{s^n \pm 1}{p^{n+1}}$, and therefore $x = \frac{s(s^n \pm 1)}{p^{n+1}}$. These values of x and y written in the given expression present it in the form $\left(\frac{s^n(s^n \pm 1)^n \pm (s^n \pm 1)^n}{p^{n(n+1)}}\right)^{\frac{1}{n+1}} = \frac{s^n \pm 1}{p^n}$, which is always rational for any values of s and p.

QUESTION XIV. (100.)—*By Mr. William Lenhart, Penn.*
To find when the subtangent of a cissoid is a maximum.

FIRST SOLUTION. *By Mr. James Divver, N. C. College.*
In Vince's Fluxions, Art. 23, the subtangent of a cissoid $= \frac{2a(a-x)}{3a-2x}$, the fluxion of which equated to 0, gives

$$\frac{4x^2\dot{x} - 12a\dot{x}x + 6a^2\dot{x}}{(3a-2x)^2} = 0;$$

whence $x = \frac{3a}{2} \pm \frac{a\sqrt{3}}{2}$.

The positive sign gives an imaginary value for y, as may be readily seen by substituting $\frac{3a}{2} - \frac{a\sqrt{3}}{2}$ for x in the equation of the curve, $y^2 = \frac{x^3}{a-x}$; therefore $x = \frac{3a}{2} - \frac{a\sqrt{3}}{2}$ is the value of x that will make the subtangent a maximum, because it will give the value of the *second fluxional coefficient* a negative quantity.

SECOND SOLUTION.—*By Mr. Farrell Ward, N. Y.*
The equation of the cissoid is $y^2 = \frac{x^2}{a-x}$, and from this the subtangent $\frac{y\dot{x}}{\dot{y}} = \frac{2ax - 2x^2}{3a - 2x}$; by putting the fluxion of this last expression $= 0$, dividing by $\dot{x}$, &c. we shall find $4x^2 - 12ax = -6a^2$, from which $x = \frac{3a \pm a\sqrt{3}}{2}$; but the upper sign cannot be used, as it

would give x greater than, hence $x=\frac{3a-a\sqrt{3}}{2}=$ .633974a.

THIRD SOLUTION.—*By Mr. Michael Floy, N. Y.*

The equation of the cissoid is $(a-x)\,y^2=x^3$ by taking the fluxions in order to determine the ratio of $\dot{x}$ to $\dot{y}$ we have $2a\dot{y}y-2x\dot{y}y-y^2\dot{x}=3x^2\dot{x}$, whence $\frac{\dot{x}}{\dot{y}}=\frac{2ay-2xy}{y^2+3x^2}$

and $y\frac{\dot{x}}{\dot{y}}=\frac{2ay^2-2xy^2}{y^2+3x^2}=$(by substituting the value of y^2)

$$\frac{(2a-2x)\left(\frac{x^3}{a-x}\right)}{\frac{x^3}{a-x}+3x^2}=\frac{2ax^3-2x^4}{3ax^2-2x^3}=\frac{2ax-2x^2}{3a-2x}=$$ value of sub-

tangent and $\frac{2ax-2x^2}{3a-2x}=$ max. per quest.

whence $(2a\dot{x}-4x\dot{x})\,(3a-2x)+2\dot{x}(2ax-2x^2)=0$
or $(a-2x)\,(3a-2x)+2ax-2x^2=0$
or $3a^2-8ax+4x^2+2ax-2x^2=0$
or $x^2-3ax=-\frac{3a^2}{2}$.

whence $x=\frac{3a}{2}\pm\sqrt{\left(\frac{9a^2-6a^2}{4}\right)}=\frac{3a}{2}\pm\frac{a}{2}\sqrt{3}$ value of the abscis when the subtangent is a maximum.

QUESTION XV. (101)—*By Mr. Farrell Ward, N. Y.*

Admitting the equation of a curve to be $y=r.\cos.z$, when y is the radius vector, and z the angle which it makes with the base of the curve, it is required to determine the area of the greatest rectangle inscribed in it.

FIRST SOLUTION.—*By Mr. Eugene Nulty, Philadelphia.*

The rectangular coordinates of any part in the curve corresponding to the given equation are $r\cos.z^2$ and $r\sin.z.\cos.z$; and at any other point they are $r\cos.z'^2$

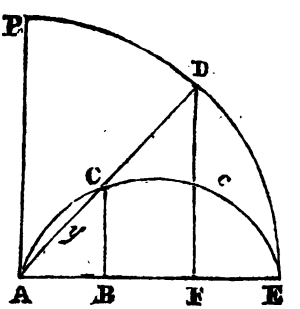
and r sin. z' cos. z'. In order that the coordinates r sin z cos. z, r sin. z' cos. z' may be equal to each other, and the inscribed figure a rectangle, we must have $z=\frac{\pi}{2}-z'$ and consequently cos. $z'=$ sin. z. Wherefore the base and altitude of this rectangle are r cos. $2z$ and r sin. $2z$ respectively, and therefore r^2 sin. $2z$. cos. $2z=\frac{r^2}{2}$ sin. $4z$ =a maximum. Wherefore $4z=\frac{\pi}{2}$, and the required area is $\frac{r^2}{2}$.

From $4z=\frac{\pi}{2}$, $z=\frac{\pi}{8}$, the base r cos. $2z=r$. cos. $\frac{\pi}{4}$, and the altitude r sin. $z=r$ cos. $\frac{\pi}{4}$; whence it follows that the inscribed figure is a square.

The equation given in this question is obviously that of the circle, but I presume a solution was required by the proposer without reference to the curve.

SECOND SOLUTION.—*By Dr. Bowditch.*

Let the abscissa AB=x, the ordinate BC=v; AC=y, angle BAC=z, and the equation of the curve $y=r$. cos. z; and as $x=y$. cos. z, $v=y$. sin. z, we get $x=r$. cos. z^2 $=\frac{1}{2}r(1+$ cos. $2z)$ $v=r$. cos. z. sin. $z=$ $\frac{1}{2}r$. sin. $2z$.



The equal values of v corresponding to the points c, c, take place when $2z$ becomes $\pi-2z$, and the corresponding values of x are $\frac{1}{2}r$ $(1+$ cos. $2z)$ and $\frac{1}{2}r(1-$ cos. $2z)$ whose difference r. cos. $2z$ being multiplied by the ordinate $v=\frac{1}{2}r$ sin. $2z$, gives

the area $\frac{1}{2}r^2$. sin. $2z$ cos. $2z=\frac{1}{4}r^2$. sin. $4z$, which is a maximum when $4z=\frac{\pi}{2}$ or $2z=\frac{\pi}{4}=45°$, in which case the ordinate $v=\frac{1}{2}r$ sin. $2z=\frac{1}{2}r\sqrt{\frac{1}{2}}$, and the abscisses are $\frac{1}{2}r(1\mp\sqrt{\frac{1}{2}})$.

THIRD SOLUTION.—*By* Ομικρον, *N. C.*

Let AB and AE (see the above Figure) be two rectangular coordinates, and on AE with radius = unity, describe a quadrant of a circle, and in it take any point D, from which demit the perpendicular DF, and join AD; then in it take AC$=r$. AF, and C will be a point in the curve. From C let fall the perpendicular CB$=y'$; then per sim. triangles, (assuming AB$=x$) $(x^2+y'^2)^{\frac{1}{2}} : x :: 1 : \frac{1}{r}(x^2+y'^2)^{\frac{1}{2}}$; whence $y'^2=rx-x^2$, an equation to the circle when radius is $\frac{r}{2}$; consequently the area of the greatest inscribed rectangle is $\frac{r^2}{2}$.

FOURTH SOLUTION.—*By Mr. Charles Wilder, Baltimore.*

Here y sin. $z=r$ cos. z sin. $z=$ altitude of the rectangle $=y'$ sin. $z'=r$ cos. z cos. z'; hence sin. $z=$sin. z' or cos. $z=-$ cos. z', and the base of the rectangle is y cos. $z-y'$ cos. $z'=r(\cos.^2z-$ cos. z cos. $z')=2r\cos.^2z$ and the rectangle $=2r^2\cos.^3 z$ sin. z which, by the question, is to be a maximum; hence 3 cos.^{2} z sin.^{2} $z dz-$ cos. $dz=0$, or 3 sin. $z=$ cos.^{2} $z=1-$sin.^{2} z or sin. $z=\frac{1}{2}$.

QUESTION XVI. (102.)—*By Professor Strong, Hamilton College.*

A given right cone has its axis vertical and its base horizontal, a heavy body moves from a certain given point on the surface of the cone with a given velocity; it is required to determine the line on the surface of the cone which it must trace out in order that its velocity, estimated in the direction of the slant height, may may be constant.

FIRST SOLUTION.—*By Dr. Adrain.*

Let R= the slant side extending from the vertex of the cone to the given point whence the curve begins, r= the slant side to any other point in the required curve, v= the initial velocity at the beginning of the curve, v= the velocity at the extremity of r, a= the uniform velocity with which r increases, ϕ= the angle described by r on the surface of the cone, and g= the measure of common gravity reduced to the direction of a slant side.

The element of the curve being $\sqrt{(dr^2+r^2d\phi^2)}$, which is described with the velocity v in the same time that dr is described with the velocity a; therefore $\frac{v^2}{a^2}=\frac{dr^2+r^2d\phi^2}{dr^2}$, whence $ad\varphi=\frac{dr}{r}\sqrt{(v^2-a^2)}$.

By the known laws of falling bodies we have $v^2-v^2=2g(r-R)$, and therefore by substitution

$$ad\varphi=\frac{dr}{r}\sqrt{(2gr+v^2-a^2-2gR)}$$

which is the differential equation of the curve sought. To integrate this equation put $2gr+v^2-a^2-2gR=u^2$, and $v^2-a^2-2gR=\pm b^2$, and by substitution we have

$$ad\phi=\frac{2u^2du}{u^2\pm b^2},$$

which is equivalent to either of the equations

$$ad\phi=2du-\frac{2b^2du}{u^2+b^2}, \text{ or } ad\varphi=2du+\frac{2b^2du}{u^2-b^2}$$

of which the integrals are

$a\phi=2u-2b$ tan. $^{-1}\frac{u}{b}+c$, and $a\varphi=2u+b$ hyp. log. $\frac{u-b}{u+b}+c'$, in which equations we have only to substitute for u its value in terms of r, and determine the arbitrary constants c and c', so as to agree with the initial values of ϕ and r, and we shall have the equation of the curve.

When $b^2=v^2-a^2-2gR=0$, we have $ad\varphi=2du$, and therefore $a\phi=2u+c=2\sqrt{(2gr)}+c$: now as the origin of φ is arbitrary, let the initial value of $\phi=\frac{\sqrt{2gR}}{a}$, and c will be nought, and therefore $a\phi=2\sqrt{(2gr)}$ from which

if $c=\frac{a^2}{8g}$, we obtain $r=c.\varphi^2$, which is in this case the equation of the curve.

In 1687 the celebrated German philosopher Leibnitz proposed the problem of the *Isochronous curve*, which consisted in determining the curve in a vertical plane, along which a heavy body would descend freely by its gravity, so as to approach the horizon uniformly. In the present elegant question of Professor Strong, the surface on which the body moves is conical, and the condition of the motion consists in this—that the moving body must approach the horizon with an uniform vertical motion; and consequently the path required forms an *Isochronous curve.*

SECOND SOLUTION.—*By Mr. Eugene Nulty.*

Let r be a ray or slant line drawn from the vertex of the cone to any point in the curve described on the surface, s the length of the curve passed over by the body in the time t, and put $a=$ the cosine of the angle which the slant side of the cone forms with the axis, and g the force of gravity.

The velocity in the direction of the curve is $\frac{ds}{dt}$, and the velocity in the direction of the slant height is $\frac{dr}{dt}$; wherefore by dynamics and the question, we have

$$\frac{ds^2}{dt^2}=2ag(r+m) \text{ and } \frac{dr}{dt}=n,$$

in which m is an invariable quantity depending on the given velocity and the portion of the point from which the motion commences, and n is the constant velocity in the direction of the slant height of the cone.

Eliminating dt^2 from the first of these equations, by virtue of the second, we get $\frac{ds^2}{dr^2}=\frac{2ag}{n^2}(r+m)$; and consequently $\frac{ds}{dr}=\frac{\sqrt{(2ag)}}{n}.\sqrt{(r+m)}$, from which by integration there is found

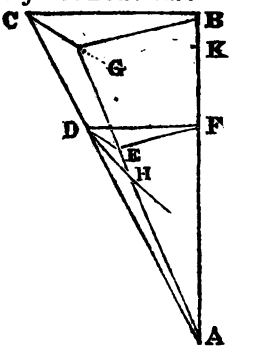
$$s+c=\frac{2}{3n}.\sqrt{(2ag)}.(r+m)^{\frac{3}{2}},$$

the equation of the curve in terms of its length s and the slant ray r.

The arbitrary constant c may be easily determined by putting $s=0$, when $r=r'$ a given quantity corresponding to the origin of motion.

By expressing ds^2 in terms of rectangular and polar coordinates, I have found other equations of the curve described by the body; but those equations are not more simple than that just given, and exhibit nothing remarkable in their forms to induce me to present their investigation.

THIRD SOLUTION.—*By Dr. Bowditch.*



Let AFB be the axis of the cone, and suppose the body at any time to be at the point D, and in the time dt, to describe the infinitely small angle DH. Through the vertex A draw the lines ADC, AHEG intersecting the base of the cone in the points CG. Let the plane DFE be parallel to the base CBG.

Put AF$=x$, FD$=y$, AD$=z$, DH$=ds$, angle DFE$=$CBG$=d\varpi$, and let the velocity at D be such as would be acquired by a body in falling freely through the space KF, from a given point K. And if we put $g=16\frac{1}{12}$ feet, we shall have $v=\sqrt{(4g\times\text{KF})}$. If we put angle CAB$=$ the half angle of the cone $=a$, and AK$=b$ cos. a, we shall have AF$=z$ cos. a, and KF$=(b-z)$ cos. a; consequently $(v=)\ \frac{ds}{dt}=\sqrt{(4g.\text{ cos. }a.)}.\sqrt{(b-z)}$. (1).

The velocity in the slant height of the cone being constant and equal to c, gives

$$-dz = cdt \quad (3)$$

The value of dt being found in this, and substituted in (1) gives

$$cds = -\sqrt{(4g.\cos a)}.\sqrt{(b-z)}.dz$$

whose integral taken so that s may commence when $z=b$, is

$$cs = \tfrac{4}{3}\sqrt{(g.\cos a)}.(b-z)^{\frac{3}{2}}$$

whence s is given in terms of z. We may also find s in terms of x, observing that $z.\cos a = x$, and if for brevity we put $h = b.\cos a = \text{AK}$; $c = \tfrac{4}{3}\sqrt{g}.\dfrac{c}{\cos. a}$.

we get $$cs = (h-x)^{\frac{3}{2}} \quad (4)$$

Taking the differential of this equation, squaring it and substituting for ds^2 its value $(x\tan a)^2.dw^2 + dx^2.\sec a^2$, and deducing the value of dw, it becomes by putting for brevity

$$m = h - \tfrac{1}{3}.c^2.\sec a^2 \qquad \text{and } \tfrac{2}{3}.c.\tan a = n$$

$$n.dw = -\frac{dx}{x}\sqrt{(m-x)}.$$

whose integral being taken so as that w may be o when $x = m$ we get

$$n.w = -2\sqrt{(m-x)} + \sqrt{(m)}\,\text{hl}\frac{\sqrt{(m)}+\sqrt{(m-x)}}{\sqrt{(m)}-\sqrt{(m-x)}}$$

and if we put $\dfrac{x}{m} = \sin u^2$, we shall have

$$w = \frac{2\sqrt{(m)}}{n}.\left\{-\cos u + \text{hl cotang } \tfrac{1}{2}u\right\}$$

Again, from the equations (1) (3) we get $-\dfrac{dz}{ds} = \dfrac{c}{\sqrt{(4g.\cos a)}}.\dfrac{1}{\sqrt{(b-z)}}$; but $-\dfrac{dz}{ds}$ represents the cosine of the angle A which the direction of the moving body makes with the line z; therefore we have

$$\cos.\ \text{A} = \frac{c}{\sqrt{(4g.\ \sin.\ a)}} \cdot \frac{1}{\sqrt{(b-z)}}.$$

Suppose at the commencement of the motion, A and z were A′ and z' respectively, in this case we could find b from A′ and z', and we should have

$$b = z' + \frac{c^2}{4g\ \cos.\ a.\ \cos.\ \text{A}'^2}.$$

Lastly, the integrals of the equation (3) gives by dividing by c

$$t = \text{constant} - \frac{z}{c},$$

or, by commencing the integral where $z = z'$, it becomes

$$t = \frac{z' - z}{c}.$$

FOURTH SOLUTION.—*By the Proposer.*

I find $ds = \sqrt{[(dzx)^2 + dx]^2}$ and $\text{V}''x\sqrt{[(dzx)^2 + dx^2]^2} = \text{V}''dx\sqrt{\left(\frac{h-\text{L}+x}{h}\right)}$ $\therefore dz = \frac{1}{c}\frac{dx}{x}\sqrt{(a+x)}$ $\left(c = \frac{\text{V}''^2 h}{\text{V}'^2}\right.$ and, $\left. a = \frac{\text{V}'^2 - \text{V}''^2}{\text{V}'^2}h - \text{L}\right)$ $\therefore z = \frac{1}{c}\left(2\sqrt{(a+x)} + \sqrt{a} \times \text{hl}\frac{\sqrt{(a+x)} - \sqrt{a}}{\sqrt{(a+x)} + \sqrt{a}}\right) + c'$ c' being the correction. In which V′ is the velocity in the direction of the curve at the commencement of the motion, and h the distance down the side of the cone which the body ought to fall to acquire V′, and V″ is the constant velocity in the direction of the side of the cone, L = distance of the point from which the body begins to move from the vertex of the cone, s = the arc of the curve described, x = the distance of the body from the vertex when it has described s, and z = the arc of a circle parallel to the base of the cone described on its surface at the distance 1 from the vertex, z being the arc of the circle intercepted by x, and the side of the cone passing through the point of departure, the correction c' is found by making $x = \text{L}$ and

$z=0$. From the above equation the curve can readily be constructed.

—

FIFTH SOLUTION.—*By Dr. Anderson.*

Put $v_{,}$ for the initial velocity of the point, c for the constant velocity in the direction of the slant height, g for the component of gravity acting in the same direction, r for the radius vector from the vertex, $r_{,}$ for the initial value of r, s for the curve from the initial position of the point, and t for the time elapsed. Then by the question and Mechanics,

$$\frac{dr}{dt}=c.$$

$$\frac{ds}{dt}=\sqrt{[v_{,}^2+2g(r-r_{,})]}$$

whence $cds=dr\sqrt{[v_{,}^2+2g(r-r_{,})]}$, and integrating,

$$(r+A)^3=a(s+B)^2,$$

which is the *equation of the curve required*, a being $=\frac{9c^2}{8g}$, and A and B, constants, determined by the equations

$$(r_{,}+A)^3=aB^2$$
$$3c(r_{,}+A)^2=2av_{,}B.$$

If the value of the above constants be altered correspondingly, the equation of the horizontal projection of the curve will also be

$$(r+A)^3=a(s+B)^2.$$

Again, $rd\phi=\sqrt{(ds^2-dr^2)}=\frac{dr}{c}\sqrt{[v_{,}^2+2g(r-r_{,})-c^2]}$,

which is integrable in circular arcs or logarithms, according as $v_{,}^2-2gr_{,}-c^2$ happens to be positive or negative.

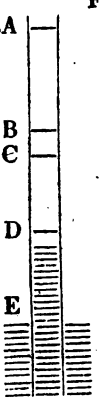
Corollary.—The rad. vector can never be less than $r_{,}-\frac{v_{,}-c^2}{2g}$, which is therefore the vector of the point to which the body might ascend if projected upward, the direction of its path being at that point directly up the conical surface. If $2gr_{,}=v_{,}^2-c^2$, that is, if $r_{,}$ is $=$ to the

altitude due to the initial horizontal velocity, then the body may ascend to the vertex; if r be less than this, the body will ascend into the upper cone.

QUESTION XVII. (103.)—*By Professor Dean.*

Suppose the pressure of the atmosphere to be equal to that of a column of water 30 feet high, and a pump to be constructed with one valve at the surface, and the other to move between 10 and 11 feet above that surface, having a motion of one foot; let the valves close perfectly tight, and open with perfect ease; how high will the action of the piston raise the water in the pump?

FIRST SOLUTION.—*By the Proposer.*



Let AE = the height of the column of water which the pressure of the atmosphere is capable of supporting $=a$

BE = the height of the working piston at its greatest elevation $=h$

BC = the extent of its motion $=c$

DE = the required height of the water $=x$

E being the place of the stationary valve at the outer surface of the water.

In the first place, it is plain that the density of the air below the piston at its lowest point, (which must be equal to that of the external air or none would escape), is to its density at its highest point inversely as the spaces which it occupies under those two positions respectively, that is, $h-x : h-c-x$: : the density of the external air $=1$: that of the air below the piston at its greatest elevation $=\frac{h-c-x}{h-x}$.

It is also well known that the water will be driven upwards by the pressure of the atmosphere, into an inverted vessel, until whatever air may be over it shall have acquired a density and pressure equal to the difference

between a and x, so that $a : a-x ::$ the density of the external air $=1$: that of the included air or density due to the height $x=\frac{a-x}{a}$. When the rarefaction effected by the ascent of the piston does not exceed this, the pressure on the internal surface of the water remains undiminished, and it will rise no further. Making this equation then $\frac{a-x}{a}=\frac{h-e-x}{h-x}$, and reducing it, $x=\frac{h\pm\sqrt{(h^2-4ae)}}{2}$.

When the given quantities are so related to each other that $4ae>h^2$, the equation is impossible—the densities which its members express can never be equal—the action of the piston will always produce a less density than is due to the column x — and the water may be raised to an indefinite height. But when $h^2>4ae$, there are two values of x, between which the prescribed action of the piston cannot produce a rarefaction equal to that which is due to the height of the column below it, and of course the ascent must stop; but if the internal surface be once raised above this interval by any other means, the piston will recommence its effect, and continue it indefinitely.

From the equation $h^2=4ae$, which forms the limit between these two cases, it is easily inferred that, in order to avoid this interruption, it is necessary that $h<2\sqrt{(ae)}$, $e>\frac{h^2}{4a}$, or $a>\frac{h^2}{4e}$. The proximity of the two valves, anywhere below A, would also effect the same purpose; but the investigation of that case would occupy more room than can be expected for it.

Substituting the numbers in the question for the symbols in the formula, we find that the water could be raised only 5 feet.

It can hardly be necessary to mention that this question was not proposed from any notion of its originality or difficulty. The principle is investigated at large by Bossut and Prony; it appears also in Gregory's Mechanics, and the Cambridge Mathematics. But it was thought that practical mechanics in this country were not aware

of the exception, and the want of this knowledge has sometimes occasioned considerable embarrassment and expense.

Second Solution.—*By Mr. Eugene Nulty, Philadelphia.*

Let h be the height of the water in the pump at the end of any number of strokes of the piston, and $h+i$ the increased height at the end of the next stroke. The pressure of the air in the pump at the commencement of this stroke is evidently the same as that of the atmosphere, and the spaces occupied by it at the beginning and end of the same stroke are $10-h$ and $11-h-i$. The pressure on the column of water $h+i$, by the nature of elastic fluids, is therefore, $30.\frac{10-h}{11-h-i}$; and since this pressure, together with the weight of the column $h+i$, must balance the pressure of the atmosphere on the water in the reservoir, we have in case of equilibrium $30\times\frac{10-h}{11-h-i}+(h+i)=30$, from which there results

$$41\,i-i^2-2hi=30-11h+h^2,$$

$$\text{or } (41-i-2h)i=(h-5).(h-6).$$

In this equation the first factor $41-i-2h$ is evidently positive for all the values of which the height h and its increment i are susceptible, and this increment is positive during the ascent of the water, vanishes when this ascent ceases, and the water remains stationary, and becomes negative only for the values of h incompatible with the circumstances attending the construction of the pump. In order, therefore, that the water may rise, both the factors $h-5$ and $h-6$ must be negative or positive, and consequently $h<5$ or >6. It must remain stationary if either of these factors vanish, and therefore $h=5$ or 6; and for all values of h between 5 and 6, the second factor $h-6$ becomes negative, and therefore the action of the piston insufficient to raise the water. Whence we infer, first, that the action of the piston cannot raise the water from the reservoir higher than 5 feet; secondly, that if the column raised to 5 feet be increased to any height between 5 and 6 feet, by pouring water in the pump, the action of the piston will be insufficient to raise

it higher ; and lastly, that if the column be further increased so as to exceed 6 feet in height, the action of the piston will raise it to the upper valve.

It has been very properly remarked by several contributors, that the theory from which the solution of this is deduced has been given by various authors. Dr. Adrain observes that in Bishop Young's Analysis of Nat. Phil. we find the general equations $x^2 - gx = (l-g)\cdot h$, which, when applied to the present example, requires us to make $g = 11$, $h = 30$, $l = 10$, and the equation becomes $x - 11x = -30$, whence $x = 5$ or 6, the less value 5 being the height to which the water will ascend, but if the pump were filled to the altitude 6, the water would remain at that height.

QUESTION XVIII. (104.)—*By Philotechnus, Philadelphia.*

Emerson in his Miscellanies finds the pressure on the axis of the wheel and axle when in motion to be $\frac{2pq(a+b)}{ap+bq}$, in which p and q represent the power and weight, and a, b, the radii of the wheel and axle ; is this expression correct ?

FIRST SOLUTION.—*By Mr. Eugene Nulty.*

Let dm be a particle of the machine at the distance r from the axis, $r\varphi$ the angle described by dm in the time t, R the pressure on the axis, and g the force of gravity.

The forces of inertia of the particle dm the power p and weight q are $dm.\frac{rd^2\phi}{dt^2}$, $pa\frac{d^2\phi}{dt^2}$ and $-qb\frac{d^2\phi}{dt^2}$. The variation of the direction in which the first of these forces acts $r\delta\varphi$, and putting $\delta\rho$ for the variation in the direction of gravity on the pressure R, the variations corresponding to the other forces are $\delta\rho + a\delta\varphi$ and $\delta\rho - b\delta\varphi$. We have therefore, by known principles,

$$\int dm.\, r\frac{d^2\varphi}{dt^2}.r\delta\varphi + pa\frac{d^2\varphi}{dt^2}.(\delta\rho + a\delta\varphi) - qb\frac{d^2\varphi}{dt^2}.(\delta\rho - b\delta\varphi) =$$
$$gp(\delta\rho + a\delta\varphi) + gq(\delta\rho - b\delta\varphi) - R\delta\rho,$$

in which the variations $\delta\varphi$ and $\delta\rho$ are independent of each other, and consequently,

$$\left\{\int dm r^2 + pa^2 + qb^2\right\}.\frac{d^2\varphi}{dt^2} = g(pa - qb),$$

$$(pa - gb)\frac{d^2\varphi}{dt^2} = g(p+q) - R$$

From these equations eliminating $\frac{d^2\varphi}{dt^2}$, we get

$$R=\frac{g(p+q).\int dmr^2+pg(a-b)^2}{\int r^2dm+pa^2+qb^2},$$

The true expression for the pressure on the axis when the inertia of the machine is taken into consideration, and which becomes simply

$$R=\frac{gpq(a-b)^2}{pa^2+qb^2},$$

when this inertia is neglected.

These are the expressions which Emerson should have found had his investigation been true and general. The pressure determined in his Miscellanies, or its equivalent given to the question, is therefore incorrect.

SECOND SOLUTION.—*By Mr. James Macully.*

Let p and q be the two weights of which p descends: put the radius of the wheel $=a$, and that of the axle $=b$; then a and b are proportional to the velocities of p and q. And let $z=$ the weight which q would sustain at p, and $x=$ the weight, which placed at p would resist the communication of rotation as much as q resists it. Then $a:b::q:z=\frac{bq}{a}$, and $x=\frac{qb^2}{a^2}$, Wood's Mechanics, Art. 354; hence $p-\frac{qb}{a}=$ force at P to move the machine, and $p+\frac{qb^2}{a^2}=$ the inertia to be moved, neglecting the inertia of the machine; hence $\frac{p-\frac{qb}{a}}{p+\frac{qb^2}{a^2}}=\frac{pa^2-qab}{pa^2+qb^2}=$ accelerating force, that of gravity being represented by unity. Again, since $\frac{pa^2-qab}{pa^2+qb^2}=$ accelerating force at P, the moving force which generates P's velocity is $\frac{pa^2-qab}{pa^2+qb^2}\times p$; therefore

$p-\frac{pa^2-qab}{pa^2+qb^2}\times p$ is that part of P's weight which is sus-

tained, or the weight which stretches the string, that is, $\frac{pqb^2+pqab}{pa^2+qb^2}$ or $\frac{(a+b).bpq}{pa^2+qb^2}$, is the weight which stretches the string AP, and consequently, $b : a :: \frac{(a+b)bpq}{pa^2+qb^2} : \frac{(a+b)apq}{pa^2+qb^2}$, and since the pressure on the centre of motion is the sum of the tensions of the strings Ap and Bq, we have $\frac{(a+b)b}{pa^2+qb^2} \times pq + \frac{(a+b)a}{pa^2+qb^2} \times pq = \frac{(a+b)^2 \times pq}{pa^2+qb^2}$ = pressure on the axis as required. Hence Emerson's formula is incorrect.

It has been remarked by various contributors, that the correct investigation and result of this question are to be found in most elementary treatises on Mechanics. For instance, Gregory, page 293, Vol. I, gives the pressure $= \frac{pq(a+b)^2}{a^2p+b^2q}$. Whewell, in his Dynamics, gives the pressure, taking the inertia of the machine into consideration, $= \frac{pq(a+b)^2+(p+q)\text{M}k^2g}{a^2p+b^2q+\text{M}k^2}$, g being gravity and $\text{M}k^2$ = moment of inertia of the machine. If this be not thus considered, then the pressure $= \frac{pq(a+b)^2}{a^2p+b^2q}$, as before.

Professor Christian Hée of Copenhagen has, in a work entitled "The pressure of weights in moving machines," investigated the above question, when the *friction* as well as the inertia of the machine is to be considered. An extract from this work is given in the Philosophical Transactions for the year 1755, and his result is

$$\text{P} = \frac{pq(a+b)^2+p\text{M}(k^2+ac\text{M})(1+ab-\frac{1}{\quad})}{a^2p+b^2q+\text{M}k^2-(1+\frac{a}{b})\text{A}ac\mu},$$

where c = radius of axis where the friction is
M = weight of the machine
k = distance of centre of forces from the centre of gravity
μ = constant ratio of friction to gravity.

QUESTION XIX. (105.)—*By Mr. John Rochford.*

Suppose a heavy body placed on the vertex of an inclined plane, and left at liberty to descend freely at the very instant that the lower end of the plane begins to be moved uniformly in a horizontal direction; required the

time of descent to the horizon, velocity acquired, and nature of the curve described, when the plane slides continually on the vertical point at a given height above the horizon; the horizontal motion being given.

FIRST SOLUTION.—*By Dr. Adrain.*

Let unity denote the height of the vertex above the given horizontal plane, a= the given uniform velocity of the inclined plane, r= any distance described from the vertex on the inclined plane, t= the time in which r is described, z= the decreasing angle contained between r and the given vertical height unity, x= tan.z, s= sec.z, g= measure of gravity.

The accelerative force of gravity in the direction of r is g cos.z, and the centrifugal force in the same direction is $r\frac{dz^2}{dt^2}$; therefore

$$(1).\ \frac{ddr}{dt^2}=r\frac{dz^2}{dt^2}+g\cos.z.$$

On account of the uniform motion, we have $-dx=adt$, whence $dt^2=\frac{dx^2}{a^2}$, also $dz=\frac{dx}{s^2}$ and cos. $z=\frac{1}{s}$, hence by substitution eq. (1) becomes, if $n=\frac{g}{a^2}$,

$$(2).\ \frac{ddr}{dx^2}=\frac{r}{s^4}+\frac{n}{s},\ \text{or}\ \frac{gddr}{dx^2}=\frac{r}{s^3}+n.$$

But, since $s^2=1+x^2$,

therefore $\frac{dds}{dx^2}=\frac{1}{s^3}$, hence by substitution eq. (2) becomes (3). $sddr-rdds=ndx^2$,

of which the integral is evidently

$$(4).\ sdr-rds=cdx+nxdx.$$

Divide eq. (4) by s^2, observing that $\frac{dx}{s^2}=dz$, and $xdx=sds$; and eq. (4) becomes

$$(5).\ \frac{sdr-rds}{s^2}=cdz+n\frac{ds}{s},$$

of which the integral is

$$(6).\ \frac{r}{s}=c'+cx+n\ \text{log.}\ s,$$

consequently,

$$(7).\ r = \text{sec. } z(c' + cz + n \text{ . log. sec. } z),$$

which is the polar equation of the curve required.

The arbitraries c and c′ are easily determined by means of the equations (4) and (7).

$$\text{Eq. (4) gives } s.\frac{dr}{dx} - r\frac{ds}{dx} = c + nx;$$

in which let $z = A =$ its primitive given value, in which case s and $\frac{ds}{dx}$ are finite, and r and $\frac{dr}{dx}$ are each $=0$; hence $0 = c + n$ tan. A, and $c = -n$ tan. A.

Again, when $z = A$ in eq. (7) we have $r = 0$, and the equation becomes

$$c' + c \text{ . } A + n \text{ log. sec. } A,$$

$$\text{whence } c' = nA. \text{ tan. } A - n \text{ log. sec. } A$$

and therefore by substituting for c and c′ their values, the equation of the curve is

$$(8)\ r = n \text{ . sec. } z\left\{(A - z) \text{ tan. } A + \text{ log. } \frac{\text{sec. } z}{\text{sec. } A}\right\}$$

when the body arrives at the horizon $r = \text{sec. } z$, and we have from eq. (8)

$$\frac{1}{n} = (A - z) \text{ tan. } A + \text{log. sec. } z - \text{log. sec. } A;$$

from which transcendental equation z may be found, and thence the value of r when the body is on the horizontal plane. And if this value of z be denoted by A′, the time in which the body arrives at the horizon will be had $= \frac{\text{tan. } A - \text{tan. } A'}{a}$.

If the body be not confined to the moving plane, it will leave the plane before the latter becomes vertical. To determine the value of the angle z when the body quits the plane, let $y = r$ cos. $z =$ the vertical descent of the body in the time t, and let the accelerative re-action of the plane be denoted by f, then f sin. $z =$ the accelerative force of f in a vertical direction upwards; consequentl $\frac{ddy}{dt^2} = g - f \text{ sin. } z,$

from which the value of the force f may be obtained for

any position of the plane; but when the body leaves the plane, $f=0$, and therefore in this case

$$\frac{ddy}{dt^2}=g, \text{ or, which is the same thing,}$$

$$\frac{ddy}{dx^2}=n.$$

But $y=r\cos. z=\frac{r}{s}=c'+cz+n\log. s$,

therefore $$\frac{dd.(c'+cz+n\log. s)}{dx^2}=n,$$

If now we perform the differentiations here indicated, (which is extremely easy, because $\frac{dz}{dx}=\frac{1}{s^2}$, and $\frac{ds}{dx}=\frac{x}{s}$,) we shall obtain the cubic equation

$$x^3+3x=2\tan. A,$$

which determines the position of the plane, and the equation (8) gives the value of the distance r when the body leaves the plane.

Second Solution.—*By Dr. Bowditch.*

Suppose the plane at the commencement of the motion to be in the situation AD and the body at A. At the end of the time t let its situation be A′B′D′ and the body at B′. Draw the horizontal lines BB′, CD′D.

Put AC$=a$, CD$=b$, AB$=x$, BB′$=y$, angle CAD$=A$, CAD′$=z$. The velocity of the point D of the plane in the direction DD′ being c, we shall have DD′$=ct$, CD$=b-ct$; hence tang. $A=\frac{b}{a}$, $\tan. z=\frac{b-ct}{a}=T$ (for brevity) and $y=xT$. Representing the force of gravity by g, we shall have, as in 199 pag. 1 Ed. La Grange's *Mec. Anal.* $\frac{d^2x}{dt^2}.\delta x+\frac{d^2y}{dt^2}.\delta y-g\delta x=0$,

But $y=xT$ gives $\delta y=\delta x.\ T$, $ddy=ddx.T+2dx.dT$, because δ does not affect t, $dt=-\frac{a}{c}\,dT$ being constant, and $ddT=0$; hence by substitution and reduction we get the differential equations (being the coefficient of δx

$$ddx(1+\mathrm{T}^2)+2\mathrm{T}d\mathrm{T}.dx-\frac{ga^2}{c^2}.\ d\mathrm{T}^2=0,$$

whose integral is $dx(1+\mathrm{T}^2)-\frac{ga^2}{c^2}\mathrm{T}d\mathrm{T}=-\frac{ga^2}{c^2}$ tan. A. $d\mathrm{T}$ the constant being taken so that $\frac{dx}{d\mathrm{T}}$ or $\frac{dx}{dt}=0$, when the body is at A and $t=0$. Dividing by $1+\mathrm{T}^2$, and integrating, it becomes

$$x=\frac{ga^2}{c^2}\left\{\text{log. }\sqrt{(1+\mathrm{TT})}-\text{tan A arc tan T}+\text{constant}\right\}$$

taking the constant so that $x=0$, when $t=0$ and $z=$A, we get the correct value of x, and as $\sqrt{(1+\mathrm{TT})}=\frac{1}{\cos. z}$, we get by reduction

$$x=\frac{ga^2}{c^2}\left\{(\mathrm{A}-z)\text{ tan. A}-\log.\frac{\cos. z}{\cos. \mathrm{A}}\right\};$$

with this and

$$y=x.\ \tan z$$

$$t=\frac{b-a\tan. z}{c}$$

we may for any value of z find the corresponding values of x, y, t; or if t be given we may find z, x, y.

THIRD SOLUTION.—*By Mr. Eugene Nulty.*

Let x and y be the vertical and horizontal coordinates of a point on the curve described, referred to two axes passing through the given point from which the body descends, a and b the given coordinates of the lower extremity of the plane at the commencement of motion or at the origin of the time t; and put $n=$ the given velocity of this extremity on the horizon, and g the force of gravity.

By Dynamics we have

$$\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y=g\delta x: \qquad (1)$$

and by the conditions of the question, the equation of the moving plane at the end of the time t, is

$$y=\frac{b-nt}{a}x. \qquad (2)$$

Eliminating y from the first of these equations by virtue of the second differentiated, we get

$$\{1+(\frac{b-nt}{a})^2\}\cdot\frac{d^2x}{dt^2}-\frac{2n}{a^2}\cdot(b-nt)\cdot\frac{dx}{dt}=g, \qquad (3)$$

of which the integral is

$$\{1+(\frac{b-nt}{a})^2\}\frac{dx}{dt}=gt. \qquad (4)$$

Let ϕ be the angle which the moveable plane when it passes through the point $(x, y,)$ forms with the vertical axis of x. We shall then have $y=x$ tan. φ, by virtue of which the equation (2) becomes

$$\text{tan.}\phi=\frac{b-nt}{a}. \qquad (5)$$

Eliminating t from the equation (4) by means of this expression differentiated and we get

$$dx=-\frac{ga}{n^2}(b-a\text{ tan. }\phi)d\varphi,$$

which integrated gives

$$x=c-\frac{ga}{n^2}(b\varphi+a\text{ log. cos. }\varphi); \qquad (6)$$

c being the perpendicular value of this integral at the origin of motion, or when tan. $\varphi=\frac{b}{a}$.

This equation determines the nature of the curve described by the body in its descent. It may be easily expressed in terms of the rectangular coordinates x and y, by means of the expression $y=x$ tan.φ; but I prefer it in the form investigated as more convenient for determining the particular value of t required in the question.

Let this value of t, or time in the which the body attains the horizon, be denoted by τ; and let β be the corresponding value of φ. Then putting a instead of φ, and β instead of φ in the equation (6), we have

$$c-a=\frac{ga}{n^2}(b\beta+a\text{ log. cos. }\beta);$$

for which determining the value of β, we shall have by means of equation (5), tan. $\beta=\frac{b-n\tau}{a}$, and consequently, the time required is

$$\tau = \frac{b - a \tan.\beta}{n}.$$

Put v = the velocity corresponding to τ just determined. The velocity of the body at the point (x, y) is $\sqrt{(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2})}$, and this by virtue of the equation (2) differentiated relatively to t, becomes

$$\sqrt{\{\frac{dx^2}{dt^2} + (\frac{b - nt}{a} \cdot \frac{dx}{dt} - \frac{nx}{a})^2\}}.$$

But when t becomes τ, the velocity $\frac{dx}{dt}$ determined from equation (4) becomes $\frac{g\tau}{1 + (\frac{b - n\tau}{a})^2} = m$, a given quantity; and x becomes equal to the given vertical coordinate a: therefore, substituting this value for $\frac{dx}{dt}$ and tan.β instead of $\frac{b - n\tau}{a}$, we have

$$V = \sqrt{\{m^2(1 + \tan.\beta^2) - 2mn \tan.\beta + n^2\}},$$

which, in conjunction with the equation (6) and the preceding value of τ, completely solves the problem.

It may not be improper to observe that the equation (6) assumes the indeterminate form $\frac{0}{0}$ when $n = 0$, and the body is supposed to fall down an inclined stationary plane. The particular value of x corresponding to this case cannot therefore be immediately deduced from this equation, but such value may be determined as follows: differentiating the numerator and denominator of equation (6), we have $-\frac{ga}{2n}(b - a \tan \phi).\frac{d\varphi}{dn} = -\frac{gat}{2} \cdot \frac{d\phi}{dn}$, by virtue of the expression $t = \frac{b - a \tan \phi}{n}$. But this value of t differentiated relatively to φ and n, gives $\frac{d\varphi}{dn} = \frac{-t}{a(1 + \tan \varphi^2)}$; wherefore by substitution $-\frac{gat}{2} \cdot \frac{d\phi}{dn}$, becomes

$\frac{gt^2}{2(1+\tan\varphi^2)}$, and putting for tan ϕ its value $\frac{b}{a}$, corresponding to the fixed position of the plane, there results $x=\frac{ga^2t^2}{2(a^2+b^2)}$, the well known expression for the case in question.

With respect to the pressure on the plane, I shall just observe that its component in the direction of x is $g-\frac{d^2x}{dt^2}$ $=g\sin\varphi\{\sin\phi+2\sin\phi\cos\varphi^2-\frac{2b}{a}\cos\varphi^2\}$ by virtue of the equations (3) (4) and (5); and consequently that its value is $g\{\sin\phi+2\sin\phi\cos\phi^2-\frac{2b}{a}\cos\phi^2\}$.

FOURTH SOLUTION.—*By Professor Strong.*

I suppose the meaning of this question is, that the body slides down the plane after the manner of a ring down a rod which passes through it, all friction being neglected; on this supposition I shall consider it. Then by a known formula of Dynamics, I have $\frac{ddr}{dt^2}=w^2r+f$, $r=$ the distance of the ring down the rod at the time, t, from the commencement of the motion, $w=$ the angular motion of the rod, and $f=$ the force of gravity down the plane. Now $w^2=\frac{h^2v^2}{(h^2+x^2)^2}$, and $f=\frac{2gh}{\sqrt{(h^2+x^2)}}$; hence we have $\frac{ddr}{dt^2}=\frac{h^2v^2r}{(h^2+x^2)^2}+\frac{2gh}{\sqrt{(h^2+x^2)}}$ in which $h=$ the height of the plane $x=$ the tangent of the angle which the rod makes with the vertical (rad. h.) and $v=$ the given horizontal velocity of the lower end of the rod, $2g=32.2$. Put $x=hz$ and we have $\frac{ddr}{dt^2}=\frac{v^2r}{h^2(1+z^2)^2}$ $+\frac{2g}{\sqrt{(1+z)^2}}$; also $dt=\frac{dx}{v}=\frac{hdz}{v}$, hence we have $\frac{ddr}{dz^2}\times\frac{v^2}{h^2}$ $=\frac{v^2}{h^2}\times\frac{r}{(1+z^2)^2}+\frac{2g}{\sqrt{(1+z)^3}}$, or $\frac{ddr}{dz^2}=\frac{r}{(1+z^2)^2}+\frac{m}{(1+z^2)^{\frac{1}{2}}}$

$\left(m=\frac{2gh^2}{v^2}\right)$ which is the differential equation of the curve sought, r being the radius vector, and $z=$ the tangent of the angle which it makes with the vertical rod (1); and it is to be noted that x^2r is the acceleration arising from the centrifugal force, and $f=$ the acceleration arising from the force of gravity down the plane, both of these manifestly accelerate the ring down the plane. The equation $\frac{d^2r}{dz^2}=\frac{r}{(1+z^2)^2}+\frac{m}{(1+z^2)^{\frac{1}{2}}}$ integrated by series gives $r=\frac{m}{2b}\times(a-z)^2\times\left(1+\frac{a\times(a-z)}{3b^2}+\frac{3a^2-b^2+1}{3\times4\times b^4}\times(a-z)^2+\frac{15a^3-9ab^2+13a}{3\times4\times5\times b^6}\times(a-z)^3+\&c.\right)$, in which $a=$ the tangent of the angle which the plane makes with the vertical at the origin of the motion, and $b=\sqrt{(1+a^2)}=$ the secant of the same angle rod (1). To determine the point where the ring meets the horizontal plane, put the above value of r equal to $h\sqrt{(1+z^2)}$, and find z from the resulting equation, which will give the tangent of the angle which the plane makes with the vertical at that time rod (1) then $\frac{(a-z')h}{v}=$ the time of motion, z' equaling the value of z, corresponding to the point where the body meets the horizontal plane, also the velocity in the direction of the curve at any point is found to be $\sqrt{\left(dr^2+\frac{r^2dz^2}{(1+z^2)^2}\right)}\frac{v}{hdz}$; (in this put for r, its value found above) and reduce, and the velocity at any point will be found in terms of z, and given quantities; then put for z its value at the horizon, and the velocity acquired will be known. Also the velocity in the direction of the plane at any point $=\frac{drv}{hdz}$ is readily found also from what has been done above. For a general method of considering questions of this kind, see Simpson's Fluxions.

FOURTH SOLUTION.—*By Dr. Anderson.*

Let the plane of the required curve (which will necessarily be vertical,) be taken for the plane of the coordinates x and y, y being vertical, and let the given fixed point be the origin. Moreover let a denote the horizontal velocity of the lower edge of the plane, b the height of the origin, c the initial distance of the plane's lower edge from its vertical position; φ the angle of depression of the plane, ϕ the initial value of φ, and t the time elapsed. Then we have

$$\cot \phi = \frac{c - at}{b} \tag{1}$$

and the equation of condition

$$y\,(c - at) - bx = L = 0. \tag{2}$$

Consequently, the general formula of Dynamics (*Mécanique Analytique*, Vol. I, p. 315) becomes, supposing the mass unity, g = accelerating force of gravity, and $\delta t = 0$ (*Méc. Anal.* p. 311).

$$\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y - g\delta y + \lambda[(c - at)\,\delta y - b\delta x] = 0,$$

λ being an indeterminate coefficient of the equation of condition. As δx and dy may now be considered independent, we obtain the two equations

$$\frac{d^2x}{dt^2} = b\lambda \tag{3}$$

$$\frac{d^2y}{dt^2} = g - \lambda\,(c - at) \tag{4}$$

Eliminating λ and substituting for $\frac{d^2x}{dt^2}$ and $(c - at)$ their values obtained from (1) and (2), we get

$$\frac{d^2y}{dt^2} = 2\frac{a}{b}\sin\phi\cos\phi\,\frac{dy}{dt} + g\sin^2\phi = 0. \tag{5}$$

To integrate this equation, let us consider $d\varphi$ constant, and t as a function of φ. Then, by the differential calculus,

$$\frac{dy}{d\varphi} = \frac{dy}{dt}\cdot\frac{dt}{d\varphi}$$

$$\frac{d^2y}{d\varphi^2} = \frac{d^2y}{dt^2}\cdot\frac{dt^2}{d\varphi^2} + \frac{dy}{dt}\cdot\frac{1}{d\varphi}d.\frac{dt}{d\varphi}$$

Substituting in this last equation for $\frac{d^2y}{dt^2}$ and $\frac{dt}{d\varphi}$ their values (1) and (5), we find after reduction

$$\frac{d^2y}{d\varphi^2} = g\frac{b^2}{a^2}\frac{1}{\sin^2\varphi}. \tag{6}$$

Integrating,

$$\frac{dy}{d\varphi} = B - A\cot\varphi \tag{7}$$

$$y = C + B\varphi - A\log.\sin\varphi. \tag{8}$$

where $A = g\frac{b^2}{a^2}$, $B = A\cot\varphi_{,}$, $C = A(\log\sin\varphi_{,} - \varphi_{,}\cot\varphi_{,})$

Equation (8) is the equation of the required curve. When the body has reached the horizon, $y = b$ and we have

$$B\varphi - A\log\sin\varphi = b - C$$

to find φ, which may be done by the common approximative methods, or by Lagrange's theorem (*Résolution des équations numeriques*. p. 229). The velocity acquired is then found by eq. (8) combined with $y = x\tan\varphi$, and $\frac{d\varphi}{dt} = \frac{a}{b}\sin^2\varphi$. The time is also found by eq. (1).

But, as the body is supposed to be only placed on the plane and not confined to it, it is necessary to ascertain whether the plane may not withdraw itself from the body before it reaches the horizon. For this purpose we must find the pressure P of the body perpendicularly to the plane. This will be (*Mèc. Anal.* Vol. II. p. 193).

$$P = \lambda\sqrt{\left\{\frac{d^2L}{dx^2} + \frac{d^2L}{dy^2}\right\}}$$

From the foregoing formulæ, this expression will be found to give

$$P = g\left\{\cos\varphi - 2\sin^3\varphi(\cot\varphi_{,} - \cot\varphi)\right\}$$

which expression being independent of a, shows that the pressure depends only on the initial position of the plane and its position at the time considered.

When the body and plane no longer act on each other, $P = 0$. Hence $\cos\varphi = 2\sin^3\varphi(\cot\varphi_{,} - \cot\varphi)$

or $\cot^3\varphi + 3\cot\varphi = 2\cot\varphi_{,}$

a cubic with one real root, namely,

$$\cot \varphi = \cot^{\frac{1}{3}} \frac{\phi}{2}, \; - \tan^{\frac{1}{3}} \frac{\phi}{2}$$

Remark. $\frac{dy}{dt} = \frac{dy}{d\varphi} \frac{d\phi}{dt} = g \frac{b^2}{a^2} (\cot \phi, - \cot \varphi) \frac{a}{b} \sin^3 \varphi = gt \sin^2 \phi$. So that the vertical velocity is always equal to that which a body would have acquired in the same time down a plane of a permanent inclination $= \varphi$.

$\frac{dy}{dt} \div \frac{d\phi}{dt} = g \frac{b}{a} t$, or the ratio of the vertical velocity to the angular, varies as the time. The vertical velocity when the plane reaches its vertical position $= g \frac{c}{a}$ varying directly as c and inversely as a.

To find when the vertical velocity is a maximum, we have

$$(\cot \phi, - \cot \varphi) \sin^2 \phi = \max$$
$$2 \cot \phi, \sin \phi \cos \phi = \cos^2 \phi - \sin^2 \varphi$$
$$\cot \phi, \sin 2\varphi = \cos 2\varphi$$
$$\tan 2\varphi = \tan \phi,$$

that is, when the plane bisects the supplement of the angle of initial elevation.

If $\phi, = 90°$, the equation to the curve becomes

$$- y = A \log \sin \varphi$$

or $$\sin \phi = e^{-\frac{y}{A}}$$

The pressure is a maximum when $6 \cos^3 \varphi - 4 \cos \varphi \cot \phi, = 3$. If $\varphi, = 90°$, $\phi = 135°$.

QUESTION XX. (106.)—*By Professor Anderson.*

Required the length and position of the principal axes or parameters of the surface, which is the *locus* of the centres of all the spheres which touch two straight given lines in position in any way in space.

FIRST SOLUTION.—*By Dr. Adrain.*

Through the straight line $2a$, which joins the given straight lines A and B at right angles, conceive a plane to pass so as to make equal angles with A and B, each of these equal angles being denoted by α: let x, y, z, be the rectangular coordinates of the centre of a sphere which touches A and B, its radius being r, the origin of the co-

ordinates being in the middle of $2a$, x being reckoned on a, y in the plane, and z at right angles to the same plane.

Since the radii r and r are at right angles to A and B, therefore by Analytical Geom. we have

$$(a+x)^2+(z\cos\alpha-y\sin\alpha)^2=r^2,$$
$$(a-x)^2+(z\cos\alpha+y\sin\alpha)^2=r^2;$$

which, by subtraction, putting $\frac{2a}{\sin 2\alpha}=b$, becomes

$$bx=yz.$$

To reduce this equation to one of the known elementary formulæ belonging to surfaces of the second degree, let us change the axes of y and z retaining that of x: this is done as usual by assuming

$$y=y'\cos\beta-z'\sin\beta,\ z=y'\sin\beta+z'\cos\beta,$$

whence $bx=(y'^2-z'^2)\cdot\sin\beta\cdot\cos\beta+y'z'\cdot(\cos^2\beta-\sin^2\beta)$, which, by taking $\cos\beta=\sin\beta=\sqrt{\frac{1}{2}}$, and putting $2b=p$, becomes, by omitting the accents,

$$px=y^2-z^2,$$

which is evidently a case of the *hyperbolic paraboloid*, the parameters of the principal parabolic sections being each equal to the given quantity $p=\frac{4a}{\sin 2\alpha}$.

The equation $px=y^2-z^2$ shows a method of describing the surface by the parallel motion of a given parabola having the parameter p, its axis being always in the plane of xy, its vertex in the curve of the parabola of which the equation is $px=y^2$, the plane of the parabola being always parallel to the plane of xz.

The equation $bx=yz$ also furnishes a description of the surface by motion. To prove this take $y=c=$ a constant quantity, and we have $bx=cz$, which is the equation of a given straight line coinciding with the surface: in like manner, taking $z=d$ we have $bx=dy$, which expresses a straight line given in position, and the axes of y and z coincide with the surface; therefore, if a straight line move along the axis of z at right angles to it, and passing through any rectilineal section made by a plane parallel to xz, it will describe the locus required.

If the perpendiculars r and r, instead of being equal, had the difference of their squares equal to a given quantity, the equation of the surface would be of the same

form as before; and if the ratio of the perpendiculars, or the sum of their squares were given, the surface would still be of the second degree.

SECOND SOLUTION.—*By Dr. Bowditch.*

Let the rectangular coordinates of the centre of the sphere be x, y, z.

The equation of the *first* of the right lines $y'=a'x'+b'$; $z'=c'x'+d'$ (1)

The equation of the *second* of the right lines $y''=a''x''+b''$, $z''=c''x''+d''$. (2)

Then the radius of the sphere being r, and the distance of its centre from these lines is r, we get

$$r^2=(x-x')^2+(y-y')^2+(z-z')^2 \quad (3)$$

$$r^2=(x-x'')+(y-y'')^2+(z-z'')^2 \quad (4)$$

Substituting the values of y', z', (1) and (3) we get for the *first line*

$$r^2=(x-x')^2+(y-a'x'-b')^2+(z-c'x'-d')^2 \quad (5)$$

and as the surface of the sphere touches the first line, this value of r^2 will not vary by changing x' into $x'+dx'$, so that we may take its differential relative to x', and put it $=0$, and dividing by $-2dx'$, we get $(x-x')+a'(y-a'x'-b')+c'(z-c'x'-d')=0$, whence

$$x'=\frac{x+a'y+c'z-a'b'-c'd'}{1+a'a'+c'c'} \quad (6)$$

Substituting this in (5), it gives r^2 in an expression of the second degree, as x, y, z, and the constants $a'b'c'd'$. In the same manner the second line gives r^2 equal to a similar expression of the second degree in x, y, z, and the constants a'', b'', c'', d''. Equating these two expressions we obtain for the *locus* of the centres of the spheres, an equation of a surface of the *second* degree in x, y, z, and a', b', c, d', a'', b'', c'', d'', whose principal axes may be found as taught in several works on these surfaces.

In particular cases this may become of the *first* degree. Thus, suppose the two given lines to be in the same plane and parallel to each other, and let the axis of x be the middle line between them. In this case $a'=0$, $a''=0$, $-b''=b'$, $c'=0$, $c''=0$, $d'=0$, $d''=0$, whence $y'=b'$, $y''=b'$, $z'=c$, $z''=c$, and equation (6) gives $x'=x$, and equation (3) gives $r^2=(y-b')^2+z^2$ for the first line, and equation (4) $r^2=(y+b')^2+z^2$. Equating these two values gives $(y-b')^2$

$=(y+b')^2$, squaring and reducing gives $b'y=0$, $y=0$, which is the equation of a plane passing through the axes x, z, which plane is the *locus* of the centres of the spheres.

If the two lines cross each other at right angles, and we take the middle line between them for the axis of x, we shall have $a'=1$ $a''=-1$, $b'=0$, $b''=0$, $c'=0$, $c''=0$, $d'=0$, $d''=0$, and the expression (6) becomes $x'=\frac{x+y}{2}$, and (5) becomes $r^2=\frac{1}{4}(x-y)^2+\frac{1}{4}(y-x)^2+z^2=\frac{1}{2}(x-y)^2+z^2$, for the first line, and for the second line $x''=\frac{x-y}{2}$, and $r^2=\frac{1}{4}(x+y)^2+z^2$, equating these values of r^2 gives $xy=0$, which is satisfied by putting $x=0$ or $y=0$, the former correspond to a plane passing through the origin perpendicular to x, the other through a plane perpendicular to y.

THIRD SOLUTION.—*By Mr. Eugene Nulty.*

Let x, y, z, be the coordinates of the centre of any of the spheres in contact with the given lines. Conceive one of these lines to be the axis of x, and let the plane of xy be parallel to the other line. Put h = the given height of this line above the plane of xy, and α= the given angle between its projection on this plane and the axis of x.

The radius of the sphere of which the centre is at the point $(x, y, z,)$ is equal to each of the perpendiculars drawn from this point to the given lines. The perpendicular to the axis of x is evidently $\sqrt{(x^2+y^2)}$, and that to the other line, supposing the origin of the coordinates to be at the intersection of the projection of this line and the axis of x, is easily seen to be $\sqrt{\{(\alpha \sin x-y \cos \alpha)^2+(h-z)^2\}}$; wherefore we have in case of contact with the given lines

$$y^2+z^2=(x \sin \alpha-y \cos \alpha)^2+(h-z)^2,$$

$$x^2-y^2-2 \cot \alpha \,.\, xy+\frac{h^2-2hz}{\sin \alpha^2}=0,$$

the equation of the locus of all the spheres in question.

To determine the parameters and the position of the principal sections corresponding to this equation, let us change the coordinates x, y, z, into other rectangular coordinates x', y', z', of which let the plane x', y' be paral-

lel to the plane xy at the distance h' above this plane, and let the new axis of x' be inclined to the projection of x on the plane $x'y'$, at an angle represented by β. We shall then have by known principles
$x=x'\cos\beta-y'\sin\beta$, $y=y'\cos\beta+x'\sin\beta$, and $z=h'-z'$, and by substituting these values, the preceding equation becomes $(x'^2-y'^2).(\cos 2\beta-\cot\alpha.\sin 2\beta)-2x'y'(\sin 2\beta+\cot\alpha.\cos 2\beta)+\frac{h^2-2hh'+2hz'}{\sin\alpha^2}=0$.

In this expression the angle β and ordinate h' are arbitrary. We may therefore assume $\sin 2\beta+\cot\alpha.\cos 2\beta=0$, and $h^2-2hh'=0$, and consequently $\tan 2\beta=-\cot\alpha=\tan(\frac{\pi}{2}+\alpha)$, $2\beta=\frac{\pi}{2}+\alpha$, $\cos 2\beta=-\sin\alpha$, $\sin 2\beta=\cos\alpha$, and $h'=\frac{h}{2}$, by virtue of which we have

$$x'^2-y'^2=\frac{2h}{\sin\alpha}z',$$

the equation of a hyperbolic paraboloid.

The coordinates x', y' and z' in this equation are fixed relatively to the given lines by the values $\beta=\frac{\pi}{4}+\frac{\alpha}{2}$, and $h'=\frac{h}{2}$ above determined, and the sections which characterise the surface are easily obtained by putting z', y', x' successively equal to determinate quantities c, b, a, for then we shall have

$$x'^2-y'^2=\frac{2hc}{\sin\alpha},\quad x'^2=\frac{2h}{\sin\alpha}z'+b^2,\quad y'^2=-\frac{2h}{\sin\alpha}x'+a^2,$$

of which the first is evidently an equation of an equilateral hyperbola, and the second and third are parabolas, having their parameters respectively equal to $\frac{2h}{\sin\alpha}$ and $-\frac{2h}{\sin\alpha}$.

If we put z', y', x' respectively, equal to nothing, we shall have

$$x'^2-y'^2=0,\quad x'^2=\frac{2h}{\sin\alpha}z' \text{ and } y'^2=-\frac{2h}{\sin\alpha}z'.$$

The first of these expressions evidently corresponds to the projections of two rectangular planes, perpendicular to the plane of xy, and assymtototes to the hyperbolic sections of which the equation is $x'^2 - y'^2 = -\frac{2h}{\sin \alpha} z'$; the second and third expressions correspond to parabolas formed by the surface and its principal sections.

As this surface is discussed in most works on Analytic Geometry I shall not take further notice of its properties, and I shall confine myself to observing, that, besides the case here considered, there are two other cases involved in the question, when taken in a general sense. One of these cases corresponds to the parallelism of the given lines, and the other to their inclination in the same plane These cases, however, seem not have been intended by the proposer, and I have considered their corresponding loci as too obvious to require formal discussion. They are evidently planes at right angles to the planes of the given lines, and bisecting the distance between those lines or their angle formed between them, according as they are supposed to be parallel or to intersect each other.

FOURTH SOLUTION.—*By the Proposer.*

Take the middle of the shortest distance $2c$ between the two given lines for the origin, the line bisecting the angle 2θ for the x axis, and the line perpendicular to both for the z axis. Then the radius r of any of the spheres will be given by either of the two equations,

$$r^2 = (x \sin \theta + y \cos \theta)^2 + (z - c)^2$$
$$r^2 = (x \sin \theta - y \cos \theta)^2 + (z + c)^2$$

whence $xy = az$, *the equation required;*

where a the *parameter* (or principal constant,) $= \frac{2c}{\sin 2\theta}$.

This is well known to be the equation of an equilateral hyperbolic paraboloid referred to its asymptotic planes; the principal axes (the two moveable straight lines which generate the surface, in their rectangular position,) coinciding with the x and y axes.

Cor. When $\theta = 0$, the lines are parallel; the parameter a of the surface becomes $= \infty$, and the surface

lel to the plane xy at the distance h' above this plane, and let the new axis of x' be inclined to the projection of x on the plane $x'y'$, at an angle represented by β. We shall then have by known principles
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Cor. When $\theta = 0$, the lines are parallel; the parameter a of the surface becomes $= \infty$, and the surface

itself degenerates to a plane ($z = 0$). When $c = 0$, the lines meet, the parameter a becomes 0, and the surface degenerates to two planes ($xy = 0$).

Remark. The above expressions for r^2, are obvious from geometrical considerations, but might have been obtained from the general expression for the shortest distance from a given point to a given straight line

$r^2 = (\eta\cos\gamma - \zeta\cos\beta)^2 + (\zeta\cos\alpha - \xi\cos\gamma)^2 + (\xi\cos\beta - \eta\cos\alpha)^2$

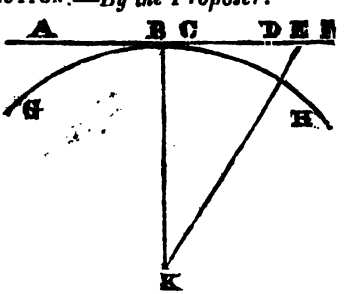
where $\xi = x - a$, $\eta = y - b$, $\zeta = z - c$, x, y, z, a, b, c, the coordinates of the given point, and a given point in the line, and α, β, γ, its angles with the axes. This expression is remarkable for its symmetry, and is not given, I believe, in the treatises on Analytical Geometry.

QUESTION XXI. (107) **OR PRIZE QUESTION.**

By Professor Adrain, Rutger's College, New Brunswick, N. J.

It is required to determine the time of the very small oscillations of an extremely slender, uniform, and inflexible rectilinear bar, placed horizontally at rest on the surface of an uniform sphere, supposing the point of contact to be exceedingly near the middle of the bar, and the gravitation of the bar towards the sphere to be according the law of nature.

FIRST PRIZE SOLUTION.—*By the Proposer.*



Let K be the centre of the sphere, and CBH an arc of a great circle in the plane of which the bar AF oscillates; let B be the point of contact at any instant of the motion, and C the middle of the bar; and let FE, ED be each equal to BC, consequently BD is equal to AB.

Let the radius BK $= r$, CF $=$ BE $= a =$ half the length of

the bar, BC = DE = EF = x, KE = $b = \sqrt{(r^2 + a^2)}$, and f = the gravity towards the sphere at B: and suppose the motion of the bar to be such that by its sliding on the point of contact B, the point C approaches B, and that by its rolling on the point of contact B this point approaches C.

The force of gravity at E towards K is $\frac{a^2 f}{b^2}$, which may be resolved into the two forces $\frac{ar^2 f}{b^3}$ and $\frac{r^3 f}{b^3}$, the former in the direction EB, the latter at right angles to EB. The indefinitely little mass $2x$ = DF being acted on by the gravity $\frac{ar^2 f}{b^3}$ in the direction EB accelerates the mass $2a$ in the same direction, and the measure of this accelerative force is $\frac{ar^2 f}{b^3} \cdot \frac{2x}{2a} = \frac{r^2 f}{b^3} \cdot x$, with which accelerative force the point C is urged towards B.

Again the mass $2x$ acted on by the force $\frac{r^3 f}{b^3}$ in a direction at right angles to BE, by means of the lever BE = a, accelerates the angular motion of the mass $2a$ about the point of contact B; reckoning the angle at the distance unity, the measure of this acceleration by the common rule given by writers on mechanics, is $\frac{2x \cdot \frac{r^3 f}{b^3} \cdot a}{\int \rho^2 d\rho}$, in which ρ is any distance from B, and $d\rho$ is the element of the mass $2a$. The denominator is in this case $\frac{2}{3}a^3$, because x is indefinitely small, and therefore the angular acceleration of AF or KB is $\frac{3r^3 f}{a^2 b^3} x$, and consequently the acceleration of the point of contact B towards C is $\frac{3r^4 f}{a^2 b^3} \cdot x$.

When x is not indefinitely small other terms arise in the acceleration, not only from the action of gravity but

also from the motion of the system; but in the present case there are none which affect the motion besides those already investigated. If therefore we put $\frac{r^2 f}{b^3} + \frac{3rf}{a^2 b^2} = \mu$, the measure of the accelerative velocity with which x is diminished is simply $\mu . x$; and therefore the forces of acceleration is as the distance, which is the usual law of vibrating bodies.

Now, let $\pi = 3.14159$ &c. and T = the time of oscillation between two successive points of rest, or between two successive transits of the middle point C of the bar through the point of contact B, and by the common rule for all such cases, the time of vibration will be determined by the equation

$$T = \frac{\pi}{\sqrt{\mu}} = \frac{\pi a b^{\frac{3}{2}}}{r f^{\frac{1}{2}} . (3r^2 + a^2)^{\frac{1}{2}}}.$$

It is obvious that the acceleration with which x decreases is nothing but $-\frac{ddx}{dt^2}$, and therefore the motion is properly defined by the equation

$$\frac{ddx}{dt^2} + \mu x = 0,$$

of which the integral may be found by various methods to be

$$x = E . \sin (t \sqrt{\mu} + \varepsilon).$$

If x and t begin together, $\varepsilon = 0$, and $x = E . \sin t\sqrt{\mu}$, and when $x = 0$, we have $\sin t \sqrt{\mu} = 0$, hence $t\sqrt{\mu}$ $= 0$ or π, 2π, 3π &c: at the end of one vibration $t =$ therefore, $T \sqrt{\mu} = \pi$, and $T = \frac{\pi}{\sqrt{\mu}}$.

The problem is resolved in the same manner and with the same facility, when the force of gravity is inversely as any power of the distance of which the index is m; we have in this general case

$$T = \left(\frac{b}{r}\right)^{\frac{m}{2}} \times \frac{\pi a b^{\frac{1}{2}}}{f^{\frac{1}{2}} . (3r^2 + a^2)^{\frac{1}{2}}} = \frac{\pi a b^{\frac{m+1}{2}}}{r^{\frac{m}{2}} . f^{\frac{1}{2}} . (3r^2 + a^2)^{\frac{1}{2}}}.$$

And in like manner we may resolve the problem, when the force of gravity is expressed by any function of the distance, or when there are several gravitating forces in different directions.

Cor. 1. When a is extremely small in comparison to r, the original equation for the value of T becomes

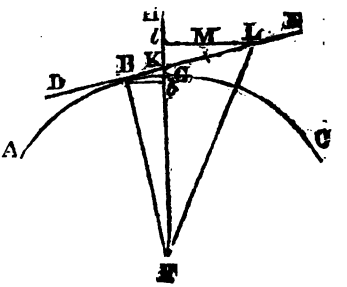
$$T = \frac{\pi a}{\sqrt{(3rf)}}.$$

Cor. 2. If we take T=1 second, and r and f the radius of the earth and the gravity at its surface, we have from the last formula $2a = 5\frac{2}{5}$ English miles, which is the length of the bar oscillating in a second of time on the surface of the earth by common gravity.

Cor. 3. Supposing, as in the preceding corollaries, that a is small in respect to r, let l be the length of vertical pendulum vibrating in the same time T at the surface of the sphere, then $T = \pi\sqrt{\frac{l}{f}}$, and thus $\frac{\pi a}{\sqrt{(3rf)}} = \pi\sqrt{\frac{l}{f}}$, whence $r = \frac{a^2}{3l}$, and thus if a and l be given, the radius of the sphere may be found.

Cor. 4. Retaining the same supposition, let $2a$ and l be two bars which vibrate at the surface of the sphere in the same time, the former horizontally the latter vertically, and we have $r = \frac{a^2}{2l}$, or $2r$ = diameter $= \frac{a^2}{l}$.

Another Solution.—*By the same.*



Let DE be the bar oscillating in the plane of the great circle of the sphere ABC, and touching it in B, and C, a point in ABC with which the middle point K of the bar comes into contact. From F the centre of the sphere, draw FG, on which from B and any other point L of the bar let fall the perpendiculars Bb, Ll; join FB, FL, and make BM = BD.

Let FG $= r$, KL $= l$, FL $= p$, Fl $= x$, Ll $= y$, angle GFB $= \varphi$ whence BG $= r\varphi$, BK $= \theta$, $z = \theta - r\varphi$, or $\theta = r\varphi + z$, gravity at G $= f$, gravity at L in LF $= P$, $m =$ a particle of the mass at L, and $t =$ the time of motion from the coincidence of K with G, so that all the four quantities t, φ θ, z, begin together: and since BL $= l + r\varphi + z$, we easily obtain the values of the rectangular coordinates x and y in terms of φ and θ, viz.

$$x = (l + r\varphi + z)\sin\varphi + r\cos\theta,$$
$$y = (l + r\varphi + z)\cos\varphi - r\sin\varphi.$$

The general equation of Lagrange for the motion of a system of bodies becomes, when the system is confined to a fixed plane,

$$(A) \qquad S\frac{ddx\delta x + ddy\delta y}{dt^2}m + S\,P\delta pm = 0,$$

in which we must substitute for x, y, p, &c., with their differentials and variations, the values in terms of the independent variables φ and z, retaining only those terms which are indefinitely greater than the others, and on which the solution depends when φ and z are indefinitely small.

Taking the differentials of the preceding equations which give the values of x and y in terms of φ and z, and retaining only those having finite coefficients, we have

$$dx = ld\varphi, \quad \text{and} \quad dy = dz,$$
$$\text{therefore } ddx = ldd\varphi, \quad \text{and} \quad ddy = ddz,$$
$$\text{and } \delta x = l\delta\varphi. \qquad \delta y = \delta z:$$

and therefore

$$ddx\delta x + ddy\delta y = \delta\varphi.\ dd\varphi.l^2 + \delta z\ ddz,$$

which, multiplied by $m = dl$ and integrated from $l = 0$ to $l = a$, we have (after doubling the result to comprehend the whole line $2a$)

$$(B)\ S\frac{\delta x ddx + \delta y ddy}{dt^2}m = \delta\varphi.\tfrac{2}{3}a^3\frac{dd\varphi}{dt^2} + \delta z.2a\frac{ddz}{dt^2}.$$

Again, to find the values of the second integral in the general formula we have $p^2 = (l+\theta)^2 + r^2$, whence $pdp = \delta\theta(l+\theta)$; and $P = \frac{r^2 f}{p^3}$, therefore $P\delta p = \delta\theta\frac{r^2 f}{p^3}.(l+\theta)$, and therefore

$$S\,P\delta pm = r^2 f d\theta.\ S\frac{(l+\theta)\,dl}{p^3}.$$

The second member of this equation must be integrated from $l-2\delta$ to $l=a$, because $ME = 2\delta$ is that part of the bar by the gravity of which the motion of the bar is affected; and δ being indefinitely small, the integral is $r^2 f \partial\delta \times \frac{2a\delta}{b^3}$, putting $b^2 = a^2 + r^2$. And because $\delta = r\varphi + z$, therefore $\partial\delta = r\delta\varphi + \partial z$; whence by substitution we have (C) $\Sigma P\delta pm = \delta\phi . \frac{2ar^3 f}{b^3} . \delta + \delta z . \frac{2ar^2 f}{b^3} . \delta$.

Substitute now instead of the two integrals in the general equation (A) their values as given by equations (B), and (C) and after division by $2a$, we shall have

$$(D). \quad \partial\varphi \left\{ \frac{a^2}{3} . \frac{dd\phi}{dt^2} + \frac{r^3 f}{b^3}\delta \right\} + \delta z \left\{ \frac{ddz}{dt^2} + \frac{r^2 f}{b^3}\delta \right\} = 0.$$

In this equation the variations, $\delta\phi$ and δz are arbitrary, and therefore the coefficients of these variations must be each = zero; and therefore putting $\frac{3r^3 f}{a^2 b^3} = \alpha, \frac{r^2 f}{b^3} = \beta$, we obtain from equation (D) the two equations following:

$$(E) \qquad \frac{dd\phi}{dt^2} + \alpha\delta = 0,$$

$$(F) \qquad \frac{ddz}{dt^2} + \beta\delta = 0.$$

To eliminate ϕ and z from these equations, multiply (A) by r, and putting $\alpha r + \beta = \mu$, we have by addition

$$(G). \qquad \frac{dd\delta}{dt^2} + \mu\delta = 0.$$

The integral of equation (G), when t and δ begin together is

$$(H) \qquad \delta = k \sin t\sqrt{\mu}.$$

in which k is the greatest value of δ.

It is evident from equation (H), in which t and δ begin at the same instant, that δ, besides being = 0 when t=0, is also = 0 when $t\sqrt{\mu}$ becomes π, 2π, 3π, &c. π being the semicircumference to the radius unity; and therefore if T = the whole time from $\delta = 0$ to $\delta = 0$ next following, we must have $T\sqrt{\mu} = \pi$, and therefore $T = \frac{\pi}{\sqrt{\mu}}$, in which, substituing for μ its value in terms of

the given quantities, we have the required value of T by the equation

$$T = \frac{\pi a b^{\frac{3}{2}}}{r f^{\frac{1}{2}} (a^2 + 3r^2)^{\frac{1}{2}}}.$$

The separate motions of the rod by sliding and rolling may be easily be determined by means of equations (E) and (F). For this purpose, multiply (F) by α, and (E) by β, and taking the difference we have

$$\alpha \frac{ddz}{dt^2} - \beta \frac{dd\varphi}{dt^2} = 0,$$

from which by integration

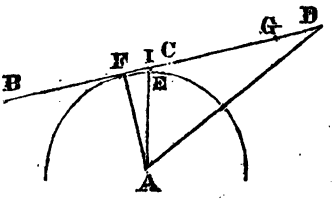
$$\alpha \frac{dz}{dt} - \beta \frac{d\varphi}{dt} = C,$$

in which C is an arbitrary constant. To obtain its value let V and V' be the rolling and sliding velocities when K coincides with G, and in that case the equation in C becomes $aV' - \frac{\beta}{r} V = C$, and if v and v' be the rolling and sliding velocities at any instant of the motion, we have

$$av' - \frac{\beta}{r} v = \alpha V' - \frac{\beta}{r} V.$$

From this equation it is evident that when C or its equal $aV' - \frac{\beta}{r} V$ is any negative quantity whatever, the velocities v and v' can never be each $= 0$ at the same instant; and in all such cases the bar can never be in a state of rest and could not therefore have been at rest at the beginniing of the motion as is given in the question. In the case therefore of the question proposed, we must have $C = 0$, and consequently $\alpha dz - \beta d\varphi = 0$, from which $\alpha z - \beta \varphi = C'$: but when K coincides with G and $t = 0$, we have $z = 0, \varphi = 0$; and therefore $C' = 0$; and thus in the case given in the question, we have $\alpha z = \beta \varphi$, from which it is manifest that the sliding and rolling velocities, as well as the spaces described by them, z and $r\varphi$ have a constant ratio, the values of these spaces being $BG = r\varphi = \frac{ark}{\mu}$. $\sin t \sqrt{\mu}$, $BK - BG = z = \frac{\beta k}{\mu}$. $\sin t \sqrt{\mu}$.

SECOND PRIZE SOLUTION.—*By Dr. Henry J. Anderson, Prof. Col. Coll. New-York.*



Let the figure represent the section made by the plane of the bar's oscillations, BD the bar extremely near the place of equilibrium, C its middle, F the point of contact, E the same in the *equilibrium* position, and $FI=FE$. Let AE, BC, AD, the sphere's accelerative force at D, FC, FI, IC, and the time be respectively denoted by $a, b, c, f, \xi, \xi', \xi''$, and t. Take $FG = BF$; then, because BG alone would be at rest on the sphere, the bar is moved by virtue of the motive force $2f\xi$ of the excess GD; rotarily by $\frac{2af\xi}{c}$, longitudinally by $\frac{2bf\xi}{c}$. Now, the effect of the first component is, by Mechanics, the same as if it took place upon a mass $= \frac{1}{3}b$ (= moment of inertia of the bar, divided by FD^2) placed at D, and the effect of the second is upon the bar's whole mass $2b$. The accelerating forces are therefore $\frac{3af}{bc}\xi$ and $\frac{f}{c}\xi$ respectively; $\frac{b}{a}\xi'$ and ξ'' are the distances due to them, which distances they tend to diminish. We have therefore

$$(1) \qquad \frac{d^2\xi'}{dt^2} + \frac{3a^2}{b^2}\frac{f}{c}\xi = 0$$

$$(2) \qquad \frac{d^2\xi''}{dt^2} + \frac{f}{c}\xi = 0$$

whence, by addition, because $\xi' + \xi'' = \xi$

$$(3) \qquad \frac{d^2\xi}{dt^2} + \frac{3a^2+b^2}{b^2}\frac{f}{c}\xi = 0.$$

Integrating from rest to rest, we get *the time required*

$$T = \pi\sqrt{\left\{\frac{b^2}{3a^2+b^2}\frac{c}{f}\right\}}.$$

Corollary I. The value of T will be given by this very same formula, whatever be the law of attraction which can be attributed to a sphere either uniform, or of a density varying as any function of the central distance.

II. It is evident that ϱ' is that part of ϱ which arises only from the rolling motion of the bar, ϱ'' that which arises only from the sliding motion. From (1) and (2) we find $\varrho' : \varrho'' :: 3a^2 : b^2$. If the rolling only is permitted, $\varrho' = \varrho$. If the sliding only, $\varrho'' = \varrho$. In these cases (1) and (2) apply separately and furnish

$$T' = \pi \sqrt{\frac{b^2}{3a^2}\frac{c}{f}}$$

$$T'' = \pi \sqrt{\frac{c}{f}}.$$

If λ, λ', λ'', denote the three corresponding isochronous pendulums, then

$$\frac{1}{\lambda} = \frac{1}{\lambda'} + \frac{1}{\lambda''}.$$

III. That there are no rotary or longitudinal sub-oscillations, will appear from the following equations derived from (1) (2) (3):

$$\begin{aligned} \varrho' &= \varepsilon' \cos kt \\ \varrho'' &= \varepsilon'' \cos kt \\ \varrho &= (\varepsilon' + \varepsilon'') \cos kt. \end{aligned}$$

where $k = \sqrt{\frac{f}{\lambda}}$, and ε' and ε'' the initial values of ϱ' and ϱ''.

IV. If a = earth's mean semidiameter, $f = 32.182$ feet, and $T = 1''$, then $2b = 5.415$ miles, the length of the bar, which placed upon the earth's surface considered perfectly spherical and smooth, would vibrate seconds.

II. Other Solutions.—*By the Same.*

Let the plane of the bar and the centre of the sphere (in which plane, it is evident, the bar will always remain) be taken for the plane of the coordinates x and y of the elements Dr of the bar; let the centre of the sphere be the origin, and a straight line parallel to the *equilibrium* position of the bar, the axis of the abscisses. Let θ denote the dip of the bar, and r the distance of Dr from the point of the bar's contact with the sphere, the rest of the notation remaining as before.

Then, evidently,

(4) $x = r\cos\theta - a\sin\theta$

(5) $y = r\sin\theta + a\cos\theta.$

Now, by d'Alembert's law and that of virtual velocities,

(6) $$S\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + P\delta p\right)Dr = 0\ ;$$

where $p = \sqrt{(x^2 + y^2)}$ and $P = fc^2p^{-2}$.

If l denote the value of r in the state of equilibrium then generally $r = l + \varrho$, and for very small oscillations $\sin\theta = \theta$, $\cos\theta = 1 - \frac{1}{2}\theta^2$.

Substituting these in (4) (5), eq. (6) becomes, because $dl = 0$, $\delta l = 0$, $Dr = Dl$,

$$S\left\{\begin{array}{l}\left\{\dfrac{d^2\varrho}{dt^2} - a\dfrac{d^2\theta}{dt^2} + \dfrac{l+\varrho}{[a^2+(l+\varrho)]^{\frac{3}{2}}}\right\}\delta\varrho \\ \quad + \left\{(a^2+l^2)\dfrac{d^2\theta}{dt^2} - a\dfrac{d^2\varrho}{dt^2}\right\}\delta\theta\end{array}\right\} Dl = 0$$

Because $\delta\varrho$ and $\delta\theta$ are the same for all the elements of the bar, they may be passed from under S, and because they are by the question arbitrary, their coefficients multiplied by Dl and integrated from $l = -b$ to $l = b$, must each be equal 0. Effecting these integrations, and neglecting the higher powers of ϱ in the developement of its function in the first coefficient, we have

$$\frac{d^2\varrho}{dt^2} - a\frac{d^2\theta}{dt^2} + \frac{f}{c}\varrho = 0$$

$$a\frac{d^2\theta}{dt^2} - \frac{3a^2}{3a^2+b^2}\frac{d^2\varrho}{dt^2} = 0$$

whence $$\frac{d^2\varrho}{dt^2} + \frac{3a^2+b^2}{b^2}\frac{f}{c}\varrho = 0$$

and $$T = \pi\sqrt{\left\{\frac{b^2}{3a^2+b^2}\frac{c}{f}\right\}}.$$ *as before.*

III. *Or thus.*

Lagrange's equations (*p.* 349 *of his chapter on small oscillations,*) become, for this question,

$$x = l + \varrho - a\theta$$

$$y = a + l\theta$$

it being allowed to neglect the higher dimensions of ϱ and θ if we develope Π in powers of ϱ. Integrating for (1), [1], &c. from $-b$ to b, the equations (p. 353) become

$$0 = \frac{d^2\varrho}{dt^2} - a\frac{d^2\theta}{dt^2} + \frac{f}{c}\varrho$$

$$0 = (a^2+\tfrac{1}{3}b^2)\frac{d^2\theta}{dt^2} - a\frac{d^2\varrho}{dt^2} \qquad \textit{as before.}$$

IV. General Solution.—For any oscillations from a state of rest, (the integrations being made accordingly).

Because $\delta\varrho$ and $\delta\theta$ are arbitrary variations, Lagrange's general formula, (*Méc. Anal. Vol. I. p.* 313) gives immediately,

$$(7) \qquad d.\frac{\delta T}{\delta d\varrho} - \frac{\delta T}{\delta\varrho} + \frac{\delta V}{\delta\varrho} = 0$$

$$(8) \qquad d.\frac{\delta T}{\delta d\theta} - \frac{\delta T}{\delta\theta} + \frac{\delta V}{\delta\theta} = 0\ ;$$

where T and V are as in the page quoted.

Substituting (4) (5) and integrating, we have

$$(9) \qquad \frac{d^2\varrho - ad^2\theta - \varrho d\theta^2}{dt^2} = S\frac{rDr}{(a^2+r^2)^{\frac{3}{2}}} \qquad \begin{cases} r=\varrho-b \\ r=\varrho+b \end{cases}$$

$$(10) \qquad d\left\{\frac{(e^2+\varrho^2)d\theta - ad\varrho}{dt}\right\} = 0$$

where $e^2 = a^2+\frac{1}{3}b^2$.

Integrating (10), we have

$$(11) \qquad \frac{d\theta}{dt} = \frac{a}{e^2+\varrho^2}\frac{d\varrho}{dt}$$

the relation between the two velocities.

Integrating again, there results

$$(12) \qquad \varrho = e\ \tan\frac{e}{a}\theta$$

the equation, (algebraic, when the coefficient of θ is rational) of the *entire* trajectory described by the middle of the bar. To find ϱ and θ from t, eliminate $d\theta$ from (9) and (11). After reduction,

$$\frac{\frac{1}{3}b^2+\varrho^2}{e^2+\varrho^2}\frac{d^2\varrho}{dt^2} + \frac{a^2\varrho}{(e^2+\varrho^2)^2}\frac{d\varrho^2}{dt^2} = S\frac{rDr}{(a^2+r^2)^{\frac{3}{2}}} \qquad \begin{cases} r=\varrho-b \\ r=\varrho+b \end{cases}$$

Integrating first with S and then with f, we get (13)

$$\frac{d\varrho}{dt} = \surd\left\{\frac{e^2+\varrho^2}{b(\frac{1}{3}b^2+\varrho^2)}\left(C+\log\frac{\varrho+b+\surd[a^2+(\varrho+b)^2]}{\varrho-b+\surd[a^2+(\varrho-b)^2]}\right)\right\}$$

from which and (14), by the method of quadratures the position of the bar may be determined for any given instant of time, from any initial position whatever.

In the case of small oscillations, (13) becomes, restoring fc^2,

$$dt\sqrt{\frac{f}{\lambda}} = -\frac{d\varrho}{\sqrt{(\varepsilon^2-\varrho^2)}},$$

whence $$t\sqrt{\frac{f}{\lambda}} = \text{arc cos}\,\frac{\varrho}{\varepsilon}$$

and $$T = \pi\sqrt{\left\{\frac{b^2}{3a^2+b^2}\,\frac{c}{f}\right\}}$$ *as before.*

Cor. I. The analysis regards the bar as always being part of a straight line in contact with the sphere, the remainder of the tangent being inflexible, but without weight. When the bar leaves the sphere, its prolongation is in contact with it.

II. The trajectory (12) remains unchanged whatever be the initial position of the bar, and whatever be the law of force, attractive or repulsive, which can belong to a sphere either of uniform density, or of equal density at equal central distances. For in this case V will not contain θ.

III. The longitudinal and rotary velocities from infinity are finite and equal respectively to

$$\frac{ce}{b}\sqrt{\frac{3f}{b}}\log\frac{c+b}{c-b}, \qquad \frac{ac}{be}\sqrt{\frac{3f}{b}}\log\frac{c+b}{c-b}.$$

V. *Or thus.*

Let x' and y' represent the coordinates of the centre of gravity or the middle of the bar, P the reaction of the sphere against the pressure of the bar ; then the formulæ (A) of *Laplace* (Méc. Cél. Vol. I. p. 71) become

$$(14) \qquad 2b\frac{d^2x'}{dt^2} = -P\sin\theta - S\frac{x}{p^3}dr$$

$$(15) \qquad -2b\frac{d^2y'}{dt^2} = -P\cos\theta + S\frac{y}{p^3}dr$$

for the motion of translation ; and formula p. 89,

$$(16) \qquad -\tfrac{2}{3}b^3\frac{d^2\theta}{dt^2} = P\varrho + S\frac{a(r-\varrho)}{p^3}dr$$

for the motion of rotation round the middle of the bar.

Eliminating P from (14) (15), we obtain after reduction and integration,

$$\cos\theta\frac{d^2x'}{dt^2}+\sin\theta\frac{d^2y'}{dt^2}+R=0, \tag{17}$$

R being the definite integral in (9).

Again, adding together (14) multiplied by y', (15) multiplied by x', and (16), and reducing by means of (4) and (5), it will be found that the right sides destroy one another, and we have this simple result,

$$\frac{y'd^2x'-x'd^2y'}{dt^2}=\tfrac{1}{3}b^2\frac{d^2\theta}{dt^2}$$

Integrating,

$$\frac{y'dx'-x'dy'}{dt}=\tfrac{1}{3}b^2\frac{d\theta}{dt} \tag{18}$$

Integrating again, and denoting by u the sectoral area reckoned from AE, we have

$$b^2\theta=6u; \tag{19}$$

So that the area described in any time by the radius vector of the middle of the bar, is always equal to one third of the area described in the same time by the half-bar in its motion of rotation.

When only extremely small oscillations are considered, eqs (17) and (19) become

$$\frac{d^2x'}{dt^2}+\frac{f}{c}(x'+a\theta)=0$$

$$b^2\theta=3ax'$$

Eliminating θ, there results

$$\frac{d^2x'}{dt^2}+\frac{3a^2+b^2}{b^2}\frac{f}{c}x'=0$$

whence $$T=\pi\sqrt{\left\{\frac{b^2}{3a^2+b^2}\cdot\frac{c}{f}\right\}}$$ *as before.*

Cor. I. Eqs. (17) and (18) may be reduced to (9) and (11) and give the general solution as before.

II. From (13) (14) and (15) the pressure $-P$ of the bar may be found in terms of ρ.

III. Eqs (12) and (18) show that (12) is the equation to an infinite number of symmetrical and equal asymptotic curves surrounding the sphere with their axes and asymptotes radiating from its centre; each asymptote making an angle $=\frac{a}{e}\pi$ with the preceding one. When $\frac{a}{e}$ is a rational fraction in its lowest terms, the asymptotes return into each other after $2e$ repetitions.

IV. The entire space between the curve and its asymptotes is equal to $\frac{ab^2}{3e}\pi$.

VI. *Or thus.*

The general problem and the proposed case may be solved with great simplicity and readiness as follows :

Because the forces (including the reaction of the sphere,) are central, the principle of areas must obtain, (*Méc. An.* I, 264) ; because they are functions of the central distance and that the conditions of the system are invariable the principle of living forces will apply. (*Méc. An.* I. 289.) These principles, in algebraic language, are, in the present question,

$$S\frac{ydx-xdy}{dt}Dr=0$$

$$S\left\{\frac{dx^2+dy^2}{2dt^2}+\Pi\right\}Dr=C$$

Substituting (4) (5), and integrating, we obtain

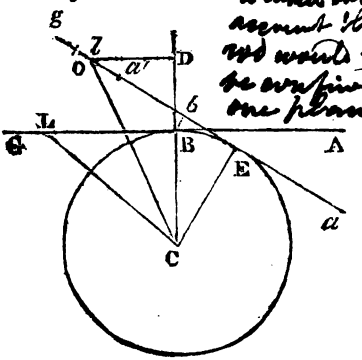
$$\frac{d\theta}{dt}=\frac{a}{e^2+\varrho^2}\frac{d\varrho}{dt}$$

$$\frac{d\varrho}{dt}=\sqrt{\left\{\frac{e^2+\varrho^2}{(\frac{1}{3}b^2+\varrho^2)}.\left(C+\log\frac{\varrho+b+\sqrt{[a^2+(\varrho+b)^2]}}{\varrho-b+\sqrt{[a^2+(\varrho-b)^2]}}\right)\right\}}$$

which are the final differential equations of the general problem, furnishing for small oscillations

$$T=\pi\sqrt{\left\{\frac{b^2}{3a^2+b^2}\frac{c}{f}\right\}} \qquad \textit{as before.}$$

THIRD SOLUTION.—*By Dr. Bowditch.*



This solution is not restricted to *small* oscillations, and it is therefore more complicated than the *second* solution which is limited. Let CBE be the magnetic ball whose centre is C, GLBE the inflexible rod resting upon it in equilibrio, on its centre of gravity B. A small oscillatory mo-

tion being given to it, let its situation at the end of the time t be $aLblg$, inclined to the horizon by the angle z. The points corresponding to the *capital* letters having taken the places marked by the same small letters respectively. Then L representing any point of the rod, which after the time t, is in the situation l, located by the vertical ordinate CD$=x$, and the horizontal ordinate D$l=y$, CB$=a$, BL$=bl=l$, BG$=$BA$=L$, C$o=f$, arch BE $=$ line $bE=az$; hence E$l=l+az$, and $cl=r=\sqrt{[a^2+(1+az)^2]}$, $x=(l+az)$ sin. $z+a$ cos. z ; $y=(l+az)$ cos. $z-a$ sin. z, whose *variations* are $\delta r=\frac{a(l+az)}{r}\delta z$, $\delta x=(l+az)$ cos $z.\ \delta z$; $\delta y=-(l+az)$ sin $z\ \delta z$.

Now the forces acting on the point l, are the gravity g, in the direction parallel to x, and the magnetic force in the direction lc or r, this last force being represented by g'. $\frac{a^2}{r^2}$, supposing the force at the surface of the ball to be g'. Hence, by La Grange's formula, (Vol. 1, pag. 195. Mec. Anal. Ed. I.) we have

$$\int\left\{\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y-g\delta x-g'.\frac{a^2}{r^2}\delta r\right\}dl=0.$$

The sign of integration $\int$ referring to the particles of the rod ag in which z is considered as constant.

Substituting the above values of dx, dy, dr, the variations are reduced to one dz and its coefficient gives,

$$\int\left\{\frac{d^2x}{dt^2}\cos z-\frac{d^2y}{dt^2}\sin z-g\cos z-g'\frac{a^3}{r^3}\right\}(l+az)dl=0.$$

Substituting the values of d^2x d^2y deduced from the above values of x, y, and neglecting the second powers of z or $\frac{dz}{dt}$, it becomes

$$-g\int(l+az)\ dl-\frac{ddz}{dt^2}\int l^2dl-g'a^3\int\frac{(l+az)}{r^3}dl=0.$$

These integrals relative to l are easily found, the last one $\int\frac{l+az}{r^3}\ dl$ being $=\int\frac{rdr}{r^3}=\int\frac{dr}{r^2}$. Taking all these integrals between the limits L$-az$ and L$+az$, corresponding to the parts a, g, it becomes by reduction and

putting for brevity $m^2=\frac{3a}{L^2}\left(g+g'\frac{a^3}{f^3}\right)$

$$\frac{ddz}{dt^2}+m^2z=0,$$

whose integral is $z=c\sin(mt+b)$, b and c being constant quantities, and if we suppose z and t to commence together in the situation of equilibrium ABG, we shall have simply,

$$z=c\sin mt.$$

c being the greatest value of z, or the angular velocity when in the situation AG multiplied by a.

ANOTHER SOLUTION.—*By the same.*

The principle of the centre of oscillation may be used to simplify considerably the above solution, when the arcs are small. Suppose for a moment the rod when in the situation ag to be acted upon *only by its gravity*, g, and let o be the centre of oscillation, b its centre of gravity, E its momentary point of suspension. Then, by the usual rule for the centre of oscillation we have $co=\frac{\int dl\,.\,El^2}{El\times 2L}=\frac{\frac{2}{3}L^3}{ax\times 2L}=\frac{L^2}{3az}$. The integral in the numerator being taken from the point a to g, or, on account of the smallness of the oscillations, from A to G. Then, by the nature of the centre of oscillation, if we divide the increment of velocity gdt of a point placed at o and falling freely by gravity, by the distance co, we shall have the increment of the angular velocity of the rod equal to $-d.\frac{dz}{dt}$. The sign — being prefixed because the velocity increases when z decreases. Hence we get $d\frac{dz}{dt}+\frac{3az}{L^2}gdt=0$, and put for brevity $m'^2=\frac{3a}{L^2}g$, it becomes $\frac{ddz}{dt^2}+m'^2z=0$, whose integral, making t and z commence together is

$$z=c\sin m't.$$

in which c represents as before the greatest value of z.

A small modification of this solution will give the value of z when the magnetic force acts also on the rod.

For if we take $\text{E}a' = \text{E}a$, it is evident that the action of both forces on the part $\text{E}a'$ is exactly balanced by the similarly situated forces acting on the part $\text{E}a$, and that forces acting on the point ga' are what produces the oscillation of the rod. Now this line ga' being very small, the forces acting on it are nearly the same as at the extreme point of the rod g or G. But the force of gravity at this point is g, and the magnetic force at that point $g'\frac{\text{CB}^2}{\text{CG}^2}$ reduced to the vertical direction is $g'\frac{\text{CB}^3}{\text{CG}^3}$ or $g'\frac{a^3}{f^3}$, the horizontal force being neglected because the line is supposed not to *slide* upon the point E or B. Therefore the vertical force acting on the extremity of the line is $g+g'\frac{a^3}{f^3}$; and as the increment of the angular velocity $-d\frac{dz}{dt}$ is proportional to this quantity, we have, to obtain the solution in this case, only to write in the preceding solution, $g + g'\frac{a^3}{f^3}$ for g, which changes m'^2 into $m^2 = \frac{3az}{\text{L}^2}\left(g+g'\frac{a^3}{f^3}\right)$ and then we shall have $\frac{ddz}{dt^2}+m^2z=0$, whose solution is as before

$$z = c \sin mt.$$

c being the greatest value of z, or the greatest angular velocity divided by m.

Let p be the length of a simple pendulum vibrating in the same time with this rod, which as in vol. I. page 31 Méc. Cél. would make $\text{T} = \pi\sqrt{(\frac{p}{g})}$. But by the above value of z, it appears that z varies from its greatest value c to its least $-c$, while the arch mt varies two right angles or π, hence $m\text{T} = \pi$; therefore, by equating these values of T, we get $p = \frac{g}{m^2}$ or $p = \frac{\text{L}^2}{3a\left(1+\frac{g'}{g}.\frac{a^3}{f^3}\right)}$, this being the length of a pendulum vibrating in the same time as the rod acted upon both by gravity and magnetism.

If we suppose the magnetic force $g'=0$, the corresponding values of p will be $p'=\frac{L^2}{3a}$, and we shall have $p' : p :: 1+\frac{g'}{g}\frac{a^3}{f^3} : 1$. If the magnetic ball be supposed to be the earth itself completely magnetized, then L being incomparably smaller than a or f, we shall have nearly $\frac{a^3}{f^3}=1$, and in this case $p' : p :: g+g' : g$. It may be observed that the length $p'=\frac{L^2}{3a}$ is equal to the distance of the centres of oscillation and gravity of the rod AG from each other, supposing the rod to be attached to the inflexible line CB void of gravity and suspended from C. If we put $L=3a$, we shall get this simple expression $p'=L$.

FOURTH PRIZE SOLUTION.—*By Mr. Eugene Nulty, Phil.*

Conceive two rectangular axes to pass through the centre of the earth in the plane in which the bar oscillates; and let x and y be the coordinates of a point in the bar, referred to these axes, m the distance of this point from the middle of the bar, x', y' the coordinates of the point in which the bar touches the earth's surface, dm a particle of the bar, dt the element of time, a the radius of the earth, and g the accelerative force of gravity on its surface.

The distance of the particle dm from the rectangular axes at the end of the time t being x and y, the forces of inertia of this particle in the instant dt are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt}$; and the attraction of the earth on the same particle for the same instant is $\frac{ga^2}{x^2+y^2}$. By known principles and the conditions of the question, we have therefore

$$\int . dm\left\{\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt}\delta y+\frac{ga^2}{x^2+y^2}.\delta\sqrt{(x^2+y^2)}\right\}=0$$

$$x'^2+y'^2=a^2, \text{ and } x'(x-x')+y'(y-y')=0,$$

of which equations, the second and third are those of the bar considered a tangent to the earth's surface at the point of contact (x', y').

Let r be the distance between the point (x', y') and the middle of the bar, and ϕ the angle which the radius a at the first of these points forms with the axes of x. The distance between this point and the point (x, y) will then be $m-r$, and the preceding equations of the bar will evidently be satisfied by $x'=a\cos\varphi$, $y'=a\sin\phi$, $x'-x=(m-r)\sin\phi$, and $y-y'=(m-r)\cos\varphi$; wherefore the coordinates of the particle dm at the point (x, y) will be $x=a\cos\phi-(m-r)\sin\phi$, and $y=a\sin\phi+(m-r)\cos\varphi$.

Differentiate these expressions and take their variations relatively to r and φ. Then put $\varphi=0$, which will simplify the results without affecting their generality, and the first of our principal equations will become by substitution and obvious reduction,

$$f.dm\left\{\left[\left((a^2+(m-r)^2\right)\frac{d^2\varphi}{dt^2}-a\frac{d^2r}{dt^2}-2(m-r)\frac{d\varphi.dr}{dt^2}\right]\delta\varphi-\left[a\frac{d^2\phi}{dt^2}\frac{d^2r}{dt^2}-(m-r)\frac{d\phi^2}{dt^2}+\frac{ga^2(m-r)}{(a^2+(m-r)^2)^{\frac{3}{2}}}\right]\delta r\right\}=0.$$

Let λ denote half the length of the bar, and let this expression be integrated relatively to m, between the limits $m=\lambda$ and $m=-\lambda$. We shall then have after dividing by 2λ.

$$\left\{\left(a^2+r^2+\frac{\lambda^2}{3}\right).\frac{d^2\phi}{dt^2}-a\frac{d^2r}{dt^2}+2r\frac{d\varphi.dr}{dt^2}\right\}\delta\varphi-\left\{a\frac{d^2\phi}{dt^2}\frac{d^2r}{dt^2}-(m-r)\frac{d\phi^2}{dt^2}\frac{ga^2}{2\lambda\sqrt{(a^2+\lambda^2)}}\left[\left(1-\frac{2\lambda r+r^2}{a^2+\lambda^2}\right)^{-\frac{1}{2}}-\left(1+\frac{2\lambda r-r^2}{a^2+\lambda^2}\right)\right]\right\}\delta r=0.$$

This equation includes all the motions of which the bar is susceptible, whatever be the distance r between its middle point and the point in which it touches the earth's surface. If we consider the variations $\delta\phi$ and δr as independent of each other, their corresponding coefficients must be separately equal to nothing, and we shall accordingly have two equations involving the sliding and rotatory motions of the bar. If we assume $\delta r=a\delta\varphi$, and consequently $r=a\varphi$, we shall satisfy the case in which the bar turns on the surface of the earth, without sliding in the direction of its length. And if we consider r an extremely small quantity, we shall be enabled to determine the times of the small oscillations in question.

In order to limit our views to these oscillations, let us neglect the square and higher powers of r, and the infinitely small term $2r\frac{drd\phi}{dt^2}$. The last term of our general equation will then become simply $\frac{ga^2r}{(a^2+\lambda^2)^{\frac{3}{2}}}$, and we shall have

$$\left\{\frac{3a^2+\lambda^2}{3}\cdot\frac{d^2\varphi}{dt^2}-a\frac{d^2r}{dt^2}\right\}\delta\varphi-\left\{a\frac{d^2\varphi}{dt^2}-\frac{d^2r}{dt^2}+r\frac{d\phi^2}{dt^2}-\frac{ga^2r}{(a^2+\lambda^2)^{\frac{3}{2}}}\right\}\delta r=0. \quad (1)$$

In this equation the arbitrary variations $\delta\phi$, δr, may be considered as independent of each other, or we may assume $\delta r=a\delta\varphi$. In the first case we shall have $\frac{3a^2+\lambda^2}{3}\cdot\frac{d^2\varphi}{dt^2}-a\frac{d^2r}{dt^2}=0$, and also

$$a\frac{d^2\varphi}{dt^2}-\frac{d^2r}{dt^2}+\frac{d\varphi^2}{dt^2}-\frac{ga^2r}{(a^2+\lambda^2)^{\frac{3}{2}}}=0,$$

and consequently, $\frac{d^2r}{dt^2}=\frac{3a^2+\lambda^2}{3a}\cdot\frac{d^2\varphi}{dt^2}$, $r=\frac{3a^2+\lambda^2}{3a}\cdot\varphi$; by virtue of which the second becomes after neglecting $r\frac{d\varphi^2}{dt^2}$,

$$\frac{d^2\varphi}{dt^2}+\frac{ga^2(3a^2+\lambda^2)}{\lambda^2(a^2+\lambda^2)^{\frac{3}{2}}}\cdot\varphi=0, \quad (2)$$

an equation by which the oscillatory motion of the bar in the case under consideration may be determined.

Again, let us put $\delta r=a\delta\phi$, and consequently $r=a\varphi$ in the equation (1). We shall then have after neglecting $r\frac{d\varphi^2}{dt^2}$,

$$\frac{d^2\varphi}{dt^2}+\frac{3ga^4}{\lambda^2(a^2+\lambda^2)^{\frac{3}{2}}}\cdot\varphi=0, \quad (3)$$

which is exactly similar to the equation (2).

Let k^2 represent a constant quantity. The equations just investigated may then be represented by

$$\frac{d^2\varphi}{dt^2}+k^2\varphi=0;$$

and, in order to determine the times of the oscillations

corresponding to them, it will be sufficient to integrate this expression and to change k^2 into the coefficients $\frac{ga^2(2a^2+\lambda^2)}{\lambda^5(a^2+\lambda^2)^{\frac{3}{2}}}$ and $\frac{3ga^4}{\lambda^5(a^2+\lambda^2)^{\frac{3}{2}}}$. Now the integral of the equation (4) is well known to be

$$\phi = c \sin(kt+c'),$$

and it only remains to determine the arbitrary constant quantities c and c'. For this purpose, let β be the value of ϕ at the commencement of motion a when $t=0$. At this instant we have $\beta = c \sin c'$, and since the velocity $\frac{d\phi}{dt}$ is then also equal to nothing, we have by differentiation, $ck \cdot \cos c' = 0$. This expression can only be satisfied by assuming $c=0$, or $\cos c'=0$, the first of which is evidently inadmissible. We have therefore $c'=\frac{\pi}{2}$, corresponding to which $\beta = c$, and our integral becomes

$$\varphi = \beta \cos kt.$$

In this equation ϕ diminishes as the angle kt increases, and becomes nothing when $kt=\frac{\pi}{2}$. As kt increases beyond this value, φ increases negatively and attains its maximum value β; when $kt=\pi$, corresponding to which the velocity $\frac{d\phi}{dt}$, which had attained its maximum the instant ϕ vanished, becomes nothing, and the bar takes a position similar to that which it had at the commencement of motion. It will therefore return towards that position, and will continue to oscillate between the limits $\phi=\beta$ and $\phi=-\beta$; its greatest velocity being βk and the time of a semi-oscillation $t=\frac{\pi}{2k}$. The times corresponding to the equations (2) and (3) are therefore

$$t=\pi \cdot \frac{\lambda(a^2+\lambda^2)^{\frac{3}{4}}}{a^2\sqrt{[g(3a^2+\lambda^2)]}} \text{ and } t=\pi \cdot \frac{\lambda(a^2+\lambda^2)^{\frac{3}{4}}}{a^2 \cdot \sqrt{(3g)}}.$$

These expressions determine the times of the small oscillations required in the question. They differ from each other in the proportion of $a\sqrt{3}$ to $\sqrt{(3a^2+\lambda^2)}$, and can be considered equal only when $\frac{\lambda}{r}$ is regarded as evanescent,

and consequently the diameter of the earth incomparably greater than the length of the bar.

*** The Solutions of Professor Strong and Augustus Shirer of Northampton to the Prize Question were remarkably able and complete. We should be happy to find in the latter gentleman a regular and permanent contributer to the Diary.

ACKNOWLEDGMENTS. &c.

The following gentlemen favoured the editor with solutions to the questions in Art. XIII. No. VI. The figures annexed to the names refer to the questions answered by each as numbered in that article.

Dr. Adrain, Rutger's College; Dr. Anderson, Columbia College; Dr. Bowditch, Boston; Professor Strong, Hamilton College; and Eugene Nulty, Philadelphia; each answered all the quesiions.

Mr. Charles Wilder, Baltimore, all but 16 and 21; Academicus of New-York, answered all but 20, 21; Nemo of New-York answered all but 16, 20, 21; Messrs. B. Mc. Gowan, Omicron N. C. and James Macully, each answered all but 16, 19, 20, 21; Charles Potts, Phil. all but 16, 18, 19, 20, 21; Farrell Ward and James Diver, S. C. College, each answered all but 13, 16, 19, 20, 21; Gerardus B. Docharty, all but 12, 15, 16, 19, 20, 21; Edward Giddings answered 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 17; James Hamilton, 1, 2, 3, 4, 5, 7, 7, 8, 10, 11, 17, 18; Mary Bond 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11; C. O. Pascalis answered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14; John Swinburne, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13; Alpheus Bixby, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 17, 18; James Sloane answered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11; William F. Kells, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14; Henry Darnall, 1, 2, 3, 4, 5, 6, 7, 13, 14, 18; D. T. Disney, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11; Michael Floy Jun'r, 1, 2, 3, 4, 5, 6, 10, 13, 14; Thomas J. Megear, 1, 2, 3, 5, 8, 9, 10. 11, 14; William J. Lewis, 1, 2, 3, 4, 5, 6, 7, 8, 11; Mathetus and James Foster each answered 1, 2, 3, 4, 5, 8, 9, 11; George Alsop 1, 2, 3, 4, 5, 6, 11; Enoch Laning, 1, 2, 3, 5, 10, 11; William Vogdes, 1, 2, 3, 4, 5, 6; William S. Denny and Robert Parry each answered 1, 2, 3, 4, 5; John Delafield Jun'r., 1, 2, 3, 4, 13; Nathan Brown 5, 8, 9, 11; Devoor V. Burger 1, 2, 3, 5; Charles Wighton, and x, y, Charleston, each answered 4; Professor Dean 17; Selah Hammond 5; Salomon Wright 6; Dennis W. Carmody 8; James Phillips 9; Joseph C. Strode 10; Charles Farquhar 13; William Lenhart 14; Philotechnus 18. Augustus Shirer, Northampton, 20 and 21.

Having found it impossible to give the preference to any one of theSolutions to the Prize Question by Dr. Adrain, Dr. Anderson, Dr. Bowditch, and Mr. Eugene Nulty, the editor has published all as prize solutions, and the PRIZE has therefore been divided equally among the above named gentlemen.

ARTICLE XV.

NEW QUESTIONS,

TO BE RESOLVED BY CORRESPONDETS IN No. VIII.

QUESTION I. (108.)—*By Mr. John D. Williams, N. Y.*

If 6 oxen or 10 colts can eat up 21 acres of pasture in 14 weeks, and 10 oxen and 6 colts can eat up 45 acres of a similar pasture in 20 weeks, the grass growing uniformly, how many sheep will eat up 240 acres in 40 weeks, admitting that 1134 sheep can eat up the same quantity as 12 oxen and 22 colts.

QUESTION II. (109.)—*By Arithmeticus, Boston.*

A person has three horses and a saddle, which of itself is worth 220 dollars; now if the saddle be put on the back of the first horse, it will make his value equal that of the second and third; but if it be put on the back of the second, it will make his value double that of the first and third; and if it be put on the back of the third, it will make his value triple that of the first and second. What is the value of each horse?

QUESTION III. (110.)—*By Mr. George Alsop.*

Given,

$$x^6y^3+x^3y^6+x^3z^3+y^3u^3=44864,$$
$$y^3+2y^4z^2+2y^4u^2+z^4=180161,$$
$$u^3+z^3=1125,$$
$$u+z=15;$$

to determine the values of x, y, z and u.

QUESTION IV. (111.)—*By John Delafield, Jun.*

Given,

$$\left.\begin{aligned} x^6+y^6&=4825,\\ x^4y+xy^4&=1092,\end{aligned}\right\}$$ to find the values of x and y.

QUESTION V. (112.)—*By Mr. John Swinburne, Brooklyn.*

Given $$\left\{\begin{aligned} x^6-2x^5y^3+y^6+x^4y-xy^4&=ax^2y^2,\\ x-y&=axy,\end{aligned}\right\}$$

to find the values of x and y.

QUESTION VI. (113.)—*By Mr. Michael Floy, N. Y.*

Required an algebraical demonstration of Question 10th, No. II. of the Diary.

QUESTION VII. (114.)—*By Mr. J. Swinburne.*

Given $\left\{\begin{array}{l} x^2+xy=3, \\ (y^2+xy)^{\frac{2}{3}}\times(y^2+xy)^{\frac{3}{2}}=1; \end{array}\right.$ to find the values of x and y.

QUESTION VIII. (115.)—*By Mr. Gerardus B Docharty.*

If through a given cone, the diameter of the base being 5 feet and perpendicular altitude 15, an equilateral triangle, whose side is 3 feet, move, having its surface parallel with the base of the cone, and its centre of gravity coinciding with the axis; What will be the solid content of the remaining part?

QUESTION IX. (116.)—*By Mr. Farrell Ward, N. Y.*

With what impetus must a perfectly elastic ball be fired from a piece of ordnance at an elevation of 42°, so that it may strike a tower standing perpendicularly to the plane of the horizon at a point 120 feet above said plane, and in rebounding from the tower and striking the horizontal plane, the sounds of both strokes shall reach the ear of the artillerist at the place of firing at the same instant of time?

QUESTION X. (117.)—*By Mary Bond.*

If the base of a triangle be divided into two parts by a straight line equal to the lesser side drawn from the vertical angle to the base; and there be given the ratio of the lesser side to its adjacent segment as 5 is to 6, the ratio of the other side to its adjacent as 17 is to 19, and the sum of the squares of the sides 389; it is required to determine the triangle?

QUESTION XI. (118.)—*By Mr. Edward Giddings.*

It is required to determine the radius of a circle, the circumference of which shall touch the arch of a quadrant of a given circle and the two semicircles described on the bounding radii of a quadrant?

QUESTION XII. (119.)—*By Mr. William Lenhart.*

Find three right angled triangles, such that the sum of the squares of the hypothenuses of two of them, may be to the sum of the squares of the base and perpendicular of the other, as 5 is to 1.

QUESTION XIII. (120.)—*By Mr. James Macully.*

Given the difference between the sun's true altitude at 8 o'clock A.M., and the latitude of the place $=17°38'30''$; to find the latitude and day of the month when the natural versed sine of double the declination is equal to the sine of the latitude.

QUESTION XIV. (121.)—*By Mr. B. Mc. Gowan.*

Find what hour on the 4th of July at the city of New-York, the variation of the sun's altitude will be a maximum?

QUESTION XV. (122.)—*By Nemo, N. Y.*

Suppose a small ball to roll down the quadrant of an ellipse by the attraction of gravitation, it is required to determine its position when its perpendicular velocity is a maximum.

QUESTION XVI. (123.)—*By Mr. Charles Wilder, Balt.*

Required the equation that if between it and $y^3+py^2+qy+r=0$, y be eliminated, the resulting equation will be of the form $x^6+mx^3+n=0$.

QUESTION XVII. 124. —*By* Ομικρον, *N. C.*

It is required to inscribe in a given paraboloid a conic frustrum whose solidity shall be a maximum.

QUESTION XVIII. (125.)—*By Mr. Siwale, Liverpool.*

Four right lines and three points are given in position; draw through one of the given points, a line meeting two of the lines given in position, so that right lines drawn from the two points of intersection to the two remaining points given in position, shall make equal angles with the two remaining lines given in position.

QUESTION XIX. (126.)—*By Professor Strong.*

It is required to draw the shortest line possible from

one given point to another on the surface of a given parabolic conoid.

QUESTION XX. (127.)—*By Mr. Eugene Nulty.*

Determine from the same expression all the small oscillations which can be made by the segment of a sphere in contact with a horizontal plane.

QUESTION XXI. (128.)—*By Dr. Anderson.*

To determine the motion of a uniform heavy inflexible circular plate, placed originally in a position nearly vertical upon a horizontal plane, and then impelled in the direction of its plane; supposing the friction just sufficient to make the plate's circumference tend to traverse without sliding, its variable projection on the horizontal plane; the resistances being as any power of the velocity for the motion edgeways, and as the square of the velocity for the motion flatways.

QUESTION XXII. (129.)—*By a Correspondent.*

Suppose a heavy body, as a point, be suspended from a fixed point by an inflexible thread void of gravity, and projected in a direction inclined to the vertical plane passing through the thread, and thus made to revolve in a spherical surface whose centre is the point of suspension. It is required to ascertain the best method of finding the situation of the body at any time, and to determine particularly the times and places corresponding to its greatest ascents and descents, and generally the horizontal angle described about the vertical which passes through the point of suspension.

QUESTION XXIII. (130.) **OR PRIZE QUESTION.**

By Mr. John Smith, Cincinnati, State of Ohio.

It is required to determine the greatest or least triangle, having its angular points on the circumferences of three circles in the same plane not meeting each other, the three circles being given both in magnitude and position.

NOTICE TO CORRESPONDENTS.

I. All communications for the Diary must be post paid, and directed to the Editor of the Mathematical Diary, 322 Broadway, New-York.

II. New Questions must be accompanied with their solutions.

III. Solutions to the Questions in No. VII, and New Questions and Solutions for No. VIII. must arrive before the 1st day of June next.

IV. No. VIII. will be published on the first day of July next.

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THE

MATHEMATICAL DIARY,

NO. VIII.

BEING THE PRIZE NUMBER OF Dr. BOWDITCH, Dr. ADRAIN, AND Mr. EUGENIUS NULTY.

ARTICLE XIV.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XV. No. VII.

QUESTION I. (108.)—*By Mr. John D. Williams, N. Y.*

If 6 oxen or 10 colts can eat up 21 acres of pasture in 14 weeks, and 10 oxen and 6 colts can eat up 45 acres of a similar pasture in 20 weeks, the grass growing uniformly, how many sheep will eat up 240 acres in 40 weeks, admitting that 1134 sheep can eat up the same quantity as 12 oxen and 22 colts.

FIRST SOLUTION.—*By Mr. Thomas J. Megear, Wilmington, Delaware.*

If 10 colts eat 21 acres of pasture in 14 weeks, 10 oxen and 6 colts, or $\frac{68}{3}$ colts will eat 68 acres in 20 weeks; but $\frac{68}{3}$ colts eat but 45 acres in that time: therefore in 6 weeks 45 acres becomes, by the growth of the grass, equal to 68 acres, and consequently 23 acres is the increase in that time: hence, as 6 : 26 : : 23 : $\frac{299}{3}$ = the increase in

26 weeks upon 45 acres, and by proportion 531 $\frac{2}{3}$ acres will be the increase upon 240 acres in that time; which added to 240, makes 771 $\frac{2}{3}$ acres for the whole quantity of pasture which is to feed a certain number of sheep 40 weeks. By proportion we find that 1134 sheep or 42 colts in 40 weeks eat 252 acres: hence, if 252 acres serve 1134 sheep, 771 $\frac{2}{3}$ acres will serve 3472 sheep.

SECOND SOLUTION.—*By Mr. Silas Warner, Wnightstown, Penn.*

In the first place reduce them all to colts; thus, if 6 oxen $=$10 colts; 10 oxen$+$6 colts$=\frac{68}{3}$ colts, and it also appears that 1 colt is equivalent to 27 sheep. Then, if 10 colts in 14 weeks eat 21 acres of grass, by proportion $\frac{150}{3}$ colts in 14 weeks, or 15 colts in 20 weeks, or $\frac{15}{2}$ colts in 40 weeks, would eat up 45 acres, provided the grass did not grow any after the first 14 weeks. But the difference $\frac{68}{3} - \frac{45}{3} = \frac{23}{3}$; that is, what grows in 20 — 14$=$6 weeks on 45 acres is sufficient to keep $\frac{23}{3}$ colts 20 weeks; or, which is the same, $\frac{23}{6}$ colts, 40 weeks. Then, as 6 : 26 : : $\frac{23}{6}$: $\frac{299}{18}$ colts, (or if what grows on 45 acres in 6 weeks will keep $\frac{23}{6}$ colts; how many colts will (40 — 14) 26 weeks' growth suffice for) that added to $\frac{15}{2}$, the number it would keep if the grass did not grow any after the first 14 weeks, makes $\frac{217}{9}$ the whole number 45 acres would keep 40 weeks. Then, as 45 : 240 : : $\frac{217}{9}$: $\frac{3472}{27}$ the number of

colts 240 acres would keep 40 weeks, which is equivalent to 3472 sheep, the number required.

QUESTION II. (109.)—*By Arithmeticus, Boston.*

A person has three horses, and a saddle, which of itself is worth 220 dollars; now if the saddle be put on the back of the first horse, it will make his value equal to that of the second and third; but if it be put on the back of the second, it will make his value double that of the first and third; and if it be put on the back of the third, it will make his value triple that of the first and second. What is the value of each horse?

FIRST SOLUTION.—*By Mr. John B. Moreau, N. Y.*

Suppose 20, 40, 60; and 30, 50, 80; and, by the rule of Double Position, proceeding as if only one condition was to be fulfilled, we find 90, 110, 200; which fulfil the first condition. Again, suppose 10, 70, 120, and 20, 80, 160; and proceeding as before, we shall find 30, 90, and 160; which will also fulfil the first condition: now, proceeding with these two sets of numbers, as if the second condition only was to be fulfilled, we find the following numbers, $6\frac{4}{9}$, $82\frac{2}{9}$, $144\frac{4}{9}$, which fulfil the first and second conditions. By proceeding in a similar manner we shall find another set of numbers, which will also fulfil the first and second conditions; for instance, 2, 76, 146. Now, proceeding with these two sets of numbers, $6\frac{4}{9}$, $82\frac{2}{9}$, $144\frac{4}{9}$, and 2, 76, 146; and by fulfilling the third condition only, we shall find the answer to be 20, 100, and 140.

SECOND SOLUTION.—*By Robert Parry, Mullica Hill.*

Let x, y, and z, denote the respective values of the horses; thus, by the conditions of the question, we derive the following equations:

$$x+220 = y+z,$$
$$y+220 = 2x+2z,$$
$$z+220 = 3x+3y.$$

Whence, $x=20$, $y=100$, and $z=140$. The answer to this question, is found in general terms, in Ryan's Algebra, page 183. Ex. 8. 2d. Ed. $x=\frac{a}{11}$, $y=\frac{5a}{11}$, and $z=$

$\frac{7a}{11}$; when a may be any number whatever: in this particular example, $a=220$; and therefore, $x=\frac{220}{11}=20$, $y=\frac{5\times220}{11}=100$, and $z=\frac{7\times220}{11}=140$, as before.

QUESTION III. (110.)—*By Mr. George Alsop.*

Given,

$$x^6y^3+x^3y^6+x^3z^2+y^3u^2=44664,$$
$$y^8+2y^4z^2+2y^4u^2+z^4=130161.$$
$$u^3+z^3=1125,$$
$$u+z=15;$$

to determine the values of x, y, z, and u.

FIRST SOLUTION.—*By the Proposer.*

Divide the third equation by the fourth, and subtract the square of the fourth from the quotient, we get $zu=50$, and $z+u=15$; whence, we find $z=5$ and $u=10$. By substituting these values in the second equation, we have $y^8+250y^4=129536$; whence $y=4$: then, by dividing the first equation by y^3, and substituting the values of y, z, and in the resulting equation, we shall have $x^6+64x^3=576$; hence $x^3=8$, and therefore $x=2$.

SECOND SOLUTION.—*By Mr. William J. Lewis, Phil.*

By cubing the fourth equation and subtracting the 3d from it, we have $3u^2z+3uz^2=225$; divide this by 3 times the third; then $uz=50$, from which and the 4th equation we find $u=10$ and $z=5$. Substituting these values in the second, we have $y^8+250y^4=129536$, from which $y=4$. Substituting in like manner in the first equation, we have $64x^6+4121x^3=37046$: hence $x=2$.

THIRD SOLUTION.—*By Mr. William Vogdes.*

From the third and fourth equations, u is readily found $=10$, and $z=5$. These values being substituted in the second equation, it becomes $y^8+50y^4+200y^4+625=180161$,*

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* Here it may be observed that those answers are correct according to the number given in the Diary, although they do not fulfil the

or $y^8+250y^4=179536$; hence $y^4=316.77$, and $y=4.219$. Substitute these values of u, y, and z in the first equation, and the value of x will be readily found by quadratics.

FOURTH SOLUTION.—*By Mr. James O'Farrell.*

From the third and fourth equations we have the sum, and sum of the cubes of two numbers given to find them. Let half the sum of the two numbers be denoted by a, and the sum of their cubes by c, and half the difference of the numbers by x; then will the numbers themselves be represented by $a-x$ and $a+x$. Therefore, will $(a-x)^3+(a+x)^3=c$, that is, by involution and reduction, $2a^3+6ax^2=c$, whence $6ax^2=c-2a^3$, and $x^2=\frac{c-2a^3}{6a}=\frac{c}{6a}-\frac{a^2}{3}$, and $x=\sqrt{\left(\frac{c}{6a}-\frac{a^2}{3}\right)}$, $=2.5$, the half difference of the numbers; to which, adding the half sum $=a=7.5$, we have $10=u$ the greater, and subtracting we have $5.=z$ the lesser. Now, substituting these values in the second original equation, we have $y^8+250y^4+625=180161$ or $y^8+250y^4=179536$. Completing the $\square$ and extracting the root $y^4+15=441.77$; extracting again $y^2=17.798$, hence $y=4.218$. Now, by substituting this value of y in the 1st original equation, the value of x will be found.

QUESTION IV. (111.)—*By John Delafield, Jun.*

Given, $\left.\begin{matrix}x^6+y^6=4825\\ x^4y+xy^4=1092\end{matrix}\right\}$ to find the values of x and y.

FIRST SOLUTION.—*By Mr. Thomas J. Megear.*

The second equation may be expressed thus, $(x^3+y^3)xy=1092$, and by squaring, $(x^6+y^6)x^2y^2+2x^5y^5=1192464$; in this equation substitute the value of x^6+y^6, and we have $2x^5y^5+4825x^2y^2=1192464$; from this equation, $xy=12$. To and from the 1st equation add and subtract $2x^3y^3$, and take the square root of each; then $x^3+y^3=91$, and $x^3-y^3=37$: hence $x^3=64$, and $y^3=27$, or $x=4$ and $y=3$.

---

condition of the question which the proposer intended; for instead of 180161, he uses 130161, as may be seen by his Solution.

SECOND SOLUTION.—*By Mr. James Divver.*

Put $x^3+y^3=s$, $x^2y^2=p$, $4825=a$, and $1092=b$; then equation 1st becomes $s^2-2p=a$, or $p=\frac{s^2-a}{2}$; from the second, $s\sqrt[3]{p}=b$; whence, $p=\frac{b^3}{s^3}$. Equating these values of p, we obtain $s^5-as^2=2b^3$; whence, by the method of divisors $s=91=x^3+y^3$. And $s^2-2p=a$, whence $p=1728=x^2+y^3=12$. Now, having the sum and the sum of the squares of two numbers, the numbers are thence easily determined, $x=3$, and $y=4$.

QUESTION V. (112)—*By Mr. John Swinburne.*

Given $\left\{\begin{array}{r} x^6-2x^3y^3+y^6+x^4y-xy^4=ax^2y^2, \\ x-y=axy, \end{array}\right\}$

to find the values of x and y.

FIRST SOLUTION.—*By Mr. James Sloan, Middleton, N. J.*

The first equation may be thus expressed, $(x^3-y^3)^2+xy\,(x^3-y^3)=ax^2y^2$; therefore, by completing the square, &c. $x^3-y^3=[\sqrt{(a+\frac{1}{4})}-\frac{1}{2}]xy=pxy$, which, divided by the second equation, gives $x^2+xy+y^2=\frac{p}{a}$; from which, subtracting the square of the second, we obtain $3xy=\frac{p}{a}-a^2x^2y^2$; whence $xy=\sqrt{\left(\frac{p}{a^3}-\frac{9}{4a^4}\right)}-\frac{3}{2a^2}=q$: hence $x+y=\sqrt{\left(\frac{p}{a}+q\right)}=m$, and $x-y=\sqrt{\left(\frac{p}{a}-29\right)}=n$; and, therefore, $x=\frac{1}{2}\,(m+n)$, and $y=\frac{1}{2}(m-n)$.

SECOND SOLUTION.—*By Mr. Henry Darnall, Phil.*

Dividing the first by the square of the second, we have $(x^2+xy+y^2)^2+\frac{xy(x^2+xy+y^2)}{x-y}=\frac{1}{a}$, or, since $\frac{xy}{x-y}=\frac{1}{a}$, $(x^2+xy+y^2)^2+\frac{1}{a}\,(x^2+xy+y^2)=\frac{1}{a}$; hence completing the square, extracting the square root, &c. $x^2+xy+y^2=-\frac{1}{2a}\pm\sqrt{\left(\frac{1}{4a^2}+\frac{1}{a}\right)}=c$; whence by combining this with the

second equation, we can readily find the values of x and y.

QUESTION VI. (113.)—*By Mr. Michael Floy, N. Y.*

Required an algebraical solution of Question 10th, No. 11. of the Diary.

FIRST SOLUTION.—*By Mr. Charles Potts, Phil.*

Let p, q, r, and s, denote respectively the right lines BD, CF, AE, and AG in the figure given with the geometrical solution, and it follows from the remarks made upon that solution that $4rp+(2s)^2=(2q)^2$. Consequently $q^2=pr+s^2$, the demonstration required.

SECOND SOLUTION.—*By Omicron, N. C.*

Let φ=semi-arc, and θ the part produced; then, $\sin.^2(\varphi+\theta)=\sin.^2\varphi\cos.^2\theta+2\sin.\varphi\cos.\varphi\sin.\theta\cos.\theta+\cos.^2\varphi\sin.^2\theta=\sin.^2\varphi\cos.^2\theta+\sin.2\varphi\cos.\theta\sin.\theta+\cos.^2\varphi\sin.^2\theta=\sin.^2\varphi-\sin.^2\varphi\sin.^2\theta+\sin.2\varphi\cos.\theta\sin.\theta+\cos.^2\varphi\sin.^2\theta\sin.^2\varphi+(\cos.^2\varphi-\sin.^2\varphi)\sin.^2\theta+\sin.2\varphi\cos.\theta\sin.\theta=\sin.^2\varphi+(\cos.2\varphi\sin.\theta+\sin.2\varphi\sin.\theta)\sin.\theta=\sin.(2\varphi+\theta)\sin.\theta+\sin.^2\varphi.$

THIRD SOLUTION.—*By Mr. Eugenius Nulty, Phil.*

Let $2a$ and b represent the arc bisected and its prolongation. Then by Trig. we have $\sin.(a+b)^2-\sin.a^2=\sin.(2a+b)-\sin.b$, and therefore

$$\sin.(2a+b)\sin.b+\sin.a^2=\sin.(a+b)^2.$$

Cor. If we write $-b$ instead of b, we shall have $\sin.(2a-b)\sin.b+\sin.(a-b)^2=\sin.a^2$; which is analogous to the prop. B. 2. Euclid's E.

QUESTION VII. (114.)—*By Mr. J. Swinburne.*

Given $\begin{cases} x^2+xy=3, \\ (y^2+xy)^{\frac{2}{3}}\times(y^2\times xy)^{\frac{2}{3}}=1 \end{cases}$; to find the values of x and y.

FIRST SOLUTION.—*By Mr. Gerardus B. Docharty.*

From the 2d of these equations $y^4+2xy^3+x^2y^2=1$, or $x^2y^2+2xy^3=1-y^4$; $\therefore$ by dividing by y^2, we have $x^2+2xy=\frac{1}{y^2}-y^2$; hence, by completing the square, &c. $x+y$

$=\frac{1}{y}$; $\therefore x=\frac{1}{y}-y$. Again, from the 1st equation, $x^2+xy=3$; completing the square, and extracting the root, $x+\frac{y}{2}=\sqrt{\left(3+\frac{y}{4}\right)}$, or $x=\sqrt{\left(3+\frac{y^2}{4}\right)}-\frac{y}{2}$. Putting these two values of $x=$ to one another, $\sqrt{\left(3+\frac{y^2}{4}\right)}-\frac{y}{2}=\frac{1}{y}-y$; $\therefore$ by transposition, $\sqrt{\left(3+\frac{y^2}{4}\right)}=\frac{1}{y}-\frac{y}{3}$; by squaring both sides $3+\frac{y^2}{4}=\frac{1}{y^2}-1+\frac{y^2}{4}$. Hence by transposition $\frac{1}{y^2}=4$; $\therefore y^2=\frac{1}{4}$, or $y=\frac{1}{2}$; and consequently $x=\frac{3}{2}$.

Second Solution.—*By Mr. John D. Williams, N. Y.*

Since $(y^2+xy)^{\frac{2}{3}}\times(y^2+xy)^{\frac{1}{2}}=(y^2+xy)^{\frac{1}{6}}=1$, $y^2+xy=1^{\frac{6}{1}}=1$; $\therefore x=\frac{1}{y}-y$. Substituting for x this value in the first equation, we get $\frac{1}{y^2}-2+y^2+1-y^2=3$; this reduced, gives $y=\frac{1}{2}$; consequently $x=\frac{3}{2}$.

Third Solution.—*By Mr. John Delafield, Jun.*

First $(y^2+xy)^{\frac{2}{3}}\times(y^2+xy)^{\frac{1}{2}}=(y^2+xy)^{\frac{1}{6}}=1$, or $y^2+xy=1$; adding this to the first equation, we have $x^2+2xy+y^2=4$, or $x+y=2$, and $x=2-y$. This substituted for x in $y^2+xy=1$, we get $y^2+2y-y^2=1$, or $2y=1$, or $y=\frac{1}{2}$, and $x=2-y=2-\frac{1}{2}=1\frac{1}{2}$, as required.

Fourth Solution.—*By Mr. James Maginness, Jun. Harrisburg, Penn.*

Here, in the second equation $(y^2+xy)^{\frac{2}{3}}\times(y^2+xy)^{\frac{1}{2}}=(y^2+xy)^{\frac{1}{6}}=1$, or $y^2+xy=1$; $\therefore x+y=\frac{1}{y}$; and from the first equation $x+y=\frac{3}{x}$; therefore, $\frac{3}{x}=\frac{1}{y}$, or $x=3y$; then substituting this value in the first equation, we have $3y^2+3y^2=12y^2=3$, or $y^2=.25$; $\therefore y=.5$,

and $x=3y=2\times.5=1.5$.

QUESTION VIII. (115.)—*By Mr. Gerardus B. Docharty.*

If through a given cone the diameter of the base being 5 feet and perpendicular altitude 15, an equilateral triangle, whose side is 3 feet, move, having its surface parallel with the base of the cone, and its centre of gravity coinciding with the axis: What will be the solid content of the remaining part?

FIRST SOLUTION.—*By Mr. M. O'Shannessy.*

The centre of gravity of an equilateral triangle, the centre of its inscribed and circumscribed circles coincide, the radius of the first is $\frac{1}{2}\sqrt{(3)}$ and of the 2d, $\sqrt{(3)}$. Now, suppose the centre at the vertex of the cone to commence moving parallel to the cones base after passing along its axis $3\sqrt{(3)}$ feet, the sides of the triangle will just touch the cone's surface, and in its motion further they will cut off 3 equal and similar conical ungulas until the angular points of the triangle just reach the cone's surface, which will be at a further distance along the axis of $3\sqrt{(3)}$ feet; again, in the triangle's progress along the axis it will cut a triangular prism out of the lower frustum of the cone, whose height is $15-6\sqrt{(3)}$ feet, and top diameter $2\sqrt{(3)}$ feet. Let its content $=f$ and that of the prism, (base $=$ the given triangle and the same height)$=p$. And let $u=$ the content of one of the ungulas found by the Rules in Mensuration, or as investigated in Simpson's Algebra, the plane side being perpendicular to the base, height$=3\sqrt{(3)}$ feet, and linear base $=3$ feet, we shall then have the remaining part of the cone $=f-p+3u$.

SECOND SOLUTION.—*By Mr. Benjamin Pierce,*

There are three parts of the cone to be considered; first, the upper part, of which the base is the inscribed circle of the triangle. This part is, therefore, wholly cut off. Secondly, there is the lower frustum, whose bases are the base of the cone and the circumscribed circle of the triangle; from this there is cut a triangular prism leaving a solid, the content of which is 47.56981 ft. Lastly, there remain three ungulas, the solidity of which, as found by Hutton's Mensuration, *p* 365, is 10.78067 ft. Therefore the solid content of the remaining part required is 58.35048 cubic feet.

THIRD SOLUTION.—*By Mr. James Dinver.*

Let a plane parallel to the base of the cone cut the curve surface in the points where the angular points of the moving equilateral triangle meet the surface; and the cone is thus divided into a frustum and an upper cone. Put P=the content of the conic frustum, and c= the content of the prism formed by the moving triangle; and a= the ungula cut off by one side of the triangle. Then the content required is $P-c+3a$.

The content of the hoof or segment cut off by a vertical plane is investigated in Hutton's Mensuration, 3d Edition, page 365.

QUESTION IX. (116.)—*By Mr. Farrell Ward.*

With what impetus must a perfectly elastic ball be fired from a piece of ordnance at an elevation of 42°, so that it may strike a tower standing perpendicularly to the plane of the horizon at a point 120 feet above said plane, and in rebounding from the tower and striking the horizontal plane, the sounds of both strokes shall reach the ear of the artillerist at the place of firing at the same instant of time.

FIRST SOLUTION.—*By Dr. Bowditch,*

The ball in striking the vertical tower is reflected back so as to describe a parabolic arc exactly similar and equal to that it would have described had it not been reflected, its position only will be inverted, and it will in both cases strike the earth in the same time T, in seconds counted from its commencing its motion. Let t be the time of striking the tower, $g=16\frac{1}{12}$ feet the force of gravity, $a=$ 1142 feet the velocity of sound per second, $b=120$ feet the height of the tower, s=the sine of the elevation, 42°, c=cosine 42°, and x=the velocity of the ball at the commencement of the motion.

Then the horizontal distance of the tower from the observer is cxt, the horizontal range cxT, the difference $cxT-cxt$ subtracted from cxt, gives the distance of the ball from the observer when it strikes the ground $=cx(2t-T)=D$, the distance of the point where the shot strikes the tower from the observer being $D'=\sqrt{[(cxt)^2+b^2]}$. But by the laws of projectiles $sxT-gT^2=0$; $sxt-gt^2=b$, the

first gives $x=\frac{gT}{s}$, and the second gives $T=\frac{b+gt^2}{gt}$; consequently $x=\frac{b+gt^2}{st}$; $2t-T=\frac{gt^2-b}{gt}$; $\therefore D=\frac{c}{gs}\cdot\frac{g^2t^4-b^2}{t^2}$, $D'=\sqrt{[\frac{c^2}{s^2}\cdot(gt^2+b)^2+b^2]}$: and per question $D'-D=a$ $(T-t)=\frac{ab}{gt}$. Substituting the values of D, D', we get an equation of the sixth degree in t, which solved gives t, and then we get $x=\frac{b+gt^2}{st}$, $T=\frac{b+gt^2}{gt}$.

QUESTION X. (117.)—*By Mary Bond.*

If the base of a triangle be divided into two parts by a straight line equal to the lesser side drawn from the vertical angle to the base; and there be given the ratio of the lesser side to its adjacent segment as 5 is to 6, the ratio of the other side to its adjacent segment as 17 is to 19, and the sum of the squares of the sides 389; it is required to determine the triangle?

FIRST SOLUTION.—*By Mr. Gerardus B. Docharty.*

Put AC*$=5x$, AD$=6x$; then because AC$=$CD, the perpendicular CE bisects AD. Whence AE or ED$=3x$, and by the properties of right angled triangles CE$=4x$. Let CB$=17y$, then DB$=9y$, $\therefore$ (by 47 P. E. 1 B.) $9x^2+54xy+81y^2+16x^2=289y^2$ or $25x^2+54xy=208y^2$ or $y^2-\frac{27xy}{104}=\frac{25x^2}{208}$.

Whence by Comp. $\square$ and extracting the root $y=\frac{x}{2}$ which substituted for y gives $\frac{289x^2}{4}+25x^2=389$ by the question.

By clearing of fractions $389x^2=1556$, $\therefore x^2=4$, $x=2$.

Consequently the sides of the triangle are 10 and 17, and the base 21.

SECOND SOLUTION.—*By Mr. J. Swinburne.*

Let ABC be the triangle, and CD the line meeting the

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* The figure can be readily supplied by the reader.

base=CA; hence, because the triangle ACD is isosceles, a perpendicular let fall from C on DB will bisect it in E. Now put AC=$5x$; then AD=$6x$, and CE=$4x$; also put BC=$17y$; then BD=$9y$. Whence per question,

$$(17y)^2+(5x)^2=389,$$

and (Euc. 1. 47) $(9y+3x)^2+(4x)^2=(17y)^2$:

The second equation being reduced, we have $54xy+25x^2=208y^2$. Put $x=vy$, and this equation becomes $54vy^2+25v^2y^2=208y^2$; dividing by y^2, we have $54v+25v^2=208$; therefore, $v=2$, and $x=vy=2y$; this value of x being substituted in the first equation and it becomes, after proper reduction, $389y^2=389$; $\therefore y=1$. Hence AC=10, BC=17, and AB=21.

QUESTION XI. (118.)—*Mr. Edward Giddings.*

It is required to determine the radius of a circle, the circumference of which shall touch the arch of a quadrant of a given circle and the two semicircles described on the bounding radii of a quadrant.

FIRST SOLUTION.—*By Dr. Adrain.*

In the straight line bisecting the given right angle produced beyond radius, take a distance equal to half the given radius, and let its extremity and the centre of one of the semicircles be joined; the straight line bisecting this last will intersect the first drawn straight line into the centre of the required circle. The demonstration is obvious.

SECOND SOLUTION.—*By Mr. Eugenius Nulty.*

Let the radius of the inscribed circle=p, that of the quadrant=r. In the triangle ABC, we have $AB^2-CD^2=BD^2-CD^2$, or $(r+p)\,p=(r-p).[r(1-\frac{1}{\sqrt{2}})-p]$, from which there results $\frac{p}{r}=\frac{2-\sqrt{2}}{6-\sqrt{2}}$, which may be easily constructed.

QUESTION XII. (119.)—*By Mr. William Lenhart.*

Find three right angled triangles, such that the sum of the squares of the hypothenuses of two of them may be to the sum of the squares of the base and perpendicular of the other, as 5 is to 1.

FIRST SOLUTION.—*By Mr. M. O'Shannessy.*

Let $5x$, $5y$, and $5z$ be the three hypothenuses; then $25x^2+25y^2 : 25z^2 :: 5 : 1$; and therefore $x^2+y^2=5z^2$. Now, if 5, which is equal to $4+1$, be divided into any other two squares p^2 and n^2 by the ordinary method, x^2 will be $=p^2z^2$, and $y^2=n^2z^2$. Now by the ordinary method taught in books, p^2 will be found $=\frac{4}{25}$, and $n^2=\frac{121}{25}$; then $x^2=\frac{4z^2}{25}$ and $y^2=\frac{121z^2}{25}$, where z may be taken at pleasure. But in order to find integral values of x and y, z must be assumed either 5 or a multiple of 5. If $z=5$, then $5x=10$, $5y=55$, and $5z=25$, the required hypothenuses; whence the legs of the first are 6 and 8, those of the second triangle 33 and 44, and of the third 15 and 20: giving three rational right angled triangles.

SECOND SOLUTION.—*By Mr. Benjamin Pierce.*

Let the first two triangles be $m(x^2-y^2)$, $m.2xy$, $m(x^2+y^2)$, and $n(x^2-y^2)$, $n.2xy$, $n(x^2+y^2)$; and let the third be x^2-y^2, $2xy$, x^2+y^2. Then $(m^2+n^2)(x^2+y^2)^2=5(x^2+y^2)^2$; $m^2+n^2=5=4+1$. By the formula in Bonnycastle's Algebra, $m=\frac{2r^2+2rs-2s^2}{r^2+s^2}$, $n=\frac{r^2+4rs-s^2}{r^2+s^2}$. Letting $x=2$, $y=1$, $s=1$, $r=3$, we have as triangles, 8.8, 6.6, 11; 1.2, 1.6, 2; and 3, 4, 5, or if we multiply each term by 5, we have as an answer in whole numbers, 44, 33, 55; 6, 8, 10; and 15, 20, 25.

THIRD SOLUTION.—*By Mr. Nathan Brown, Jun.*

Let $5x^2=$ the sum of the squares of the hypothenuses of the two first triangles. Then $x=$ the hypothenuse of the third. Now $5x^2$ consists of the two squares $4x^2$ and x^2. Each of these must contain two other squares. Divide $4x^2$ into $\frac{64}{25}x^2$ and $\frac{36}{25}x^2$. Then, if we make $x=5$, the hypothenuse of a known right angled triangle, the triangles are found to be 10, 8, 6; 5, 4, 3; 5, 4, 3. By giving to x other values we find corresponding triangles, as, $\frac{5}{6}$, $\frac{2}{3}$, $\frac{1}{2}$; $\frac{5}{12}$, $\frac{14}{39}$, $\frac{23}{156}$; $\frac{5}{12}$, $\frac{1}{3}$, $\frac{1}{4}$: 130, 104, 78; 65, 56, 33.

65, 63, 16, &c. If we divide $5x^2$ into the two squares $\frac{1}{25}x^2$ and $\frac{4}{25}x^2$, there will result such triangles as these, 143, 132, 55; 26, 24, 10; 65, 60, 25.

FOURTH SOLUTION.—*By Mr. Alpheus Bixby, N. Y.*

Let x, y, and z be the three hypothenuses of the three triangles, and by the question $x^2+y^2=5z^2$. Assume $x=m+2n$ and $y=2m-n$; square the two eq. and add them together, then will $x^2+y^2=5m^2+5n^2=5z^2$ divide by 5 and $m^2+n^2=z^2$; let $m=p^2-q^2$, and $n=2pq$, square these two eq. and $m^2+n^2=(p^2-q^2)^2+4p^2q^2=(p^2+q^2)^2=z^2$; hence $x=m+2n=p^2-q^2+4pq$, $y=2m-n=2(p^2-q^2)-2pq$, and $z^2=(p^2+q^2)^2$, or $z=p^2+q^2$, where p and q may be assumed at pleasure, and the three hypothenuses readily found, and each of them resolved into two squares to denote the legs of the triangle to which it belongs; thus, Let a^2=the given square and x^2 and a^2-x^2, its two parts. Then since x^2 is a square, it only remains to make a^2-x^2 a square. For which purpose let its root $=nx-a$, we shall find $\left(\frac{2an}{n^2+1}\right)^2$ and $\left(\frac{an^2-a}{n^2+1}\right)^2$ for the two squares required; when a and n may be any numbers taken at pleasure, provided n be greater than 1. See *Bonnycastle's Alg.* Ex. 4. page 228, *third New-York* Ed.

QUESTION XIII. (120.)—*By Mr. James Macully.*

Given the difference between the sun's true altitude at 8 o'clock A. M., and the latitude of the place $=17°38'30''$; to find the latitude and day of the month when the natural versed sine of double the declination is equal to the sine of the latitude.

FIRST SOLUTION.—*By Dr. Adrain.*

Let $a=17°38'30''$, x comp of the lat.; then $x-a$=the the sun's zenith distance, also, H=the given hour from noon $=60°$, and $y=$ his N. Polar distance, and we have cos. x cos. y+cos. H sin. x sin. y=cos. x cos. a+sin. x sin. a; whence since cos. H$=\frac{1}{2}$, tan. $x=2.\frac{\text{cos. } a-\text{cos. } y}{\text{sin. } y-2\text{ sin. } a}$, also per question cos. $x=1+$cos. $2y$.

From which equations we easily find $y=68°36'00''$ and $x=74°33'28''$; whence the latitude $=15°26'32''$ north, and the day of the month answering to the declination $21°24'$ N. is the 28th of May or the 16th of July.

SECOND SOLUTION.—*By Mr. Eugenius Nulty.*

Let l be the latitude, a the altitude, δ the declination, and put $l \sim a = m$; then we have sin. a = sin. l cos. m − cos. l sin. m, sin. l = 1 − cos. 2 δ; and by Spherical Trig. sin. a = sin. l sin. $\delta + \frac{1}{2}$ cos. l − cos. δ. From this and the first expression we get tan. $l = \frac{\text{cos. } \delta - 2 \text{ sin. } m}{2(\text{cos. } m - \text{sin. } \delta)}$; by means of which and the second expression, the values of δ and l may be determined.

THIRD SOLUTION.—*By Dr. Bowditch.*

Put $c=17°38'30''$, h = sun's altitude, d = sun's declination, l = latitude of the place, then the hour angle being 60°, and its cosine $=\frac{1}{2}$, we have

sin. h = sin. l. sin. $d + \frac{1}{2}$ cos. l. cos. d

sin. l = vers. 2 d = 2 sin. $d^2 = 2 - 2$ cos. d^2; $h = l \pm c$.

The second equation gives sin. $d = \sqrt{(\frac{1}{2} \text{ sin. } l)}$, cos. $d = \sqrt{(1 - \frac{1}{2} \text{ sin. } l)}$, substituting these and h in the first equation, and it becomes

sin. $(l+c)$ = sin. $l \cdot \sqrt{(\frac{1}{2} \text{ sin. } l)} + \frac{1}{2}$ cos. $l \cdot \sqrt{(1 - \frac{1}{2} \text{ sin. } l)}$,

whence by trials we may find l.

QUESTION XIV. (121.)—*By Mr. B. Mc. Gowan.*

Find what hour on the 4th of July at the city of New-York, the variation of the sun's altitude will be a maximum?

FIRST SOLUTION.—*By Professor Strong.*

The colatitude of New-York, the codeclination of the sun, (July 4th) and the coaltitude at the time sought, make a spheric triangle. The circle of declination described by the sun on the given day passes through the angle of the triangle formed by the intersection of the sun's codeclination and coaltitude. On the circle of declination let there be set off an indefinitely small given arc, t, (described in a given time) from the said angle on

the circle of declination either toward or from the meridian, suppose toward it, then through the lower extremity of t, imagine a parallel of altitude to be drawn, also through the higher extremity of t, draw a vertical circle; and there will by this means be formed an infinitely small spheric triangle, having t for its hypothenuse, and the portion of the vertical circle between the parallel of altitude and the higher extremity of t for its perpendicular, which will be the variation of the sun's altitude corresponding to t, also the base of the triangle will be that portion of the parallel of altitude intercepted by the vertical circles passing through the extremities of t. This triangle has also a right angle, formed by the vertical passing through the higher extremity of t. This triangle, in its ultimate state, may be regarded as rectilineal. Let φ (rad. (1))=the angle of the triangle opposite the variation of the sun's altitude, and let v denote the variation; then $v=t.$ sin. $\varphi=$ max. but, t is given $\therefore$ sin. $\varphi=$max. Now $\varphi=$the angle formed by the sun's coaltitude and codeclination, for they have the angle formed by the circle of declination and the vertical through the lower extremity of t, for their common complement. Let then $\varphi'=$the angle formed by said vertical and the meridian of New-York; also let L denote its colatitude, and D=the sun's codeclination, then by a known theorem in Spheric Trigonometry, I have sin. D : sin. L :: sin. φ' : sin. $\varphi=\frac{\text{sin. L}}{\text{sin. D}}\times$sin. φ' but L is invariable and D may be regarded as invariable without sensible error for the time here considered. $\therefore$ when sin. $\varphi=$max. sin. $\varphi'=$max. also, but sin. $\varphi'=$max. when $\varphi'=$a right angle $\therefore$ the vertical circle sought cuts the meridian at right angles, and the sun at the time required is on the Prime Vertical. Hence I have to solve a right angled spheric triangle, having D for its hypothenuse and L for one of its legs, and the angle at the pole included by D, and L is the angle sought, which gives the time before or after noon. Supposing that L = 49° 17′ 20″ D = 67° 3′, I find by calculation the time before or after noon to be 4 hours, 2 minutes, 5 seconds, supposing the variation of the sun's declination in the interval to be neglected.

The Problem here solved, is answered in most Treatises on Astronomy.

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SECOND SOLUTION.—*By Dr. Bowditch.*

It has been demonstrated by writers on astronomy (as for example, Cognoli, trig. page 442, Laland's, Astronomy T. 3. p. 596, Ed. 3.) that the sun moves fastest when in the prime vertical, and the usual rule for this, using the same symbols as in the last problem, is cos. t=tan. d . cot. l ; hence $t=60°30'\frac{1}{2}=4$ hrs. 2 m. 2 seconds.

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THIRD SOLUTION.—*By Mr. Nash, N. Y.*

For a satisfactory solution of this question nothing more is wanted than the true time of day, when the sun's centre bears east, or west of the place of observation. The bearing, in this solution, is supposed to be taken from the City Hall, New-York, in lat. 40° 42′ 45″ N. and Long. 74° W. on the 4th of July 1826, in the morning.

In the spherical triangle PBC, right angled at B, let PB represent the complement of the latitude, and PC the sun's polar distance at the time required, to find the angle BPC, or hour-angle, equal to the time before noon.

Log. Cotang. lat.+Log. Cotang. Pol. Dist.−10=Log. Cosine of the hour-angle BPC.

By the sun's Pol. Dist. at Greenwich, noon, the time was found approximately, the declination being reduced to the approximate time, and the calculation repeated once or twice more, the declination became 22° 55′ 31″ 13 N. whence the Pol. Dist.=67° 4′ 28″ 87.

Cotangent Lat.	40° 42′ 45″	10.0652414
Cotangent Pol. Dist.	67° 4′ 28″.7	9.6262759
Cosine of hour-angle	60° 33′ 40″.5	9.6915173
In Time	4*h.* 2*m.* 14*s.* 7	before noon.
True Time required	7*h.* 57*m.* 45*s.* 3	A. M.

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FOURTH SOLUTION.—*By a Correspondent, Lex. Ken.*

Let w=the sun's required horary distance from noon, and z=his altitude at that time ; put v=cos. w, y=sin. z, m=cos. dec.×cos. lat., n=sin. dec.×sin. lat., r=sin. (lat.−dec.). The general expression for the sun's alti-

tude will then be $y=mv+n$, and consequently $dz\frac{dy}{\sqrt{(1-y^2)}}$ $=\frac{mdv}{\sqrt{(1-(mv+n)^2)}}$. Now the differential of this (putting $ddz=0$, in order to obtain the maximum,) gives $-mdv^2\times(mv+n)=ddv\times[1-(mv+n)^2]$; but $dw=\frac{-dv}{\sqrt{(1-v^2)}}$, or by considering dw constant, $ddv=\frac{-rdv}{1-r^2}$, and by substitution we obtain $-mdv^2\times(mv+n)=\frac{-vdv^2}{1-v^2}\times[1-(mv+n)^2]$; which reduced (putting $a=1+\frac{r^2}{2mn}$) gives $v=a-\sqrt{(a^2-1)}=0.49242=\cos.\ 60°\ 30'$ nearly. Hence w ($=60°\ 30'$)$=4h.\ 2m$.

Question XV. (122.)—*By Nemo, N. Y.*

Suppose a small ball to roll down the quadrant of an ellipse by the attraction of gravitation, it is required to determine its position when its perpendicular velocity is a maximum.

First Solution.—*By Benjamin Pierce, Jun.*

Let x be the abscissa of the point required, then (Cambridge Mechanics, art. 339) $2g\left(\frac{b}{a}\sqrt{(\frac{1}{4}a^2-x^2)}\right)$ is the velocity of the ball along the curve.

Representing by v the perpendicular velocity, we have $\left(\frac{1}{4}a^2-x^2+\frac{b^2x^2}{a^2}\right)\frac{\frac{1}{4}a^2-x^2}{x^2} : \frac{b^2}{a^2}(\frac{1}{4}a^2-x^2) :: 2g\left(\frac{b}{a}\sqrt{(\frac{1}{4}a^2-x^2)}\right) : v$.

$v=\frac{2b^3gx^2\sqrt{(\frac{1}{4}a^2-x^2)}}{a^3\left(\frac{1}{4}a^2-x^2+\frac{b^2x^2}{a^2}\right)}$. But as v is to be a maximum, we obtain, $\frac{1}{8}a^4-x^4\left(\frac{b^2}{a^2}-2\right)-\frac{3}{4}a^2x^2=0$, which gives us the value of x directly.

Second Solution.—*By Dr. Bowditch.*

Let x, y be the rectangular ordinates of the ellipsis

counted from its centre, the ordinate y being vertical, the corresponding semiaxes being a, c and $ee=aa-cc$, so that $y^2=\frac{cc}{aa}(aa-xx)$ and s the arc described. Then the velocity in the direction of the curve is as $\sqrt{(y)}$, and in the vertical direction is $\frac{dy}{ds}.\sqrt{(y)}$. This, or its square, $y.\frac{dy^2}{ds^2}$ is a maximum or $\frac{ds^2}{ydy^2}$ a minimum. Substituting the value of y, this becomes $\frac{a}{c^3}.\frac{a^4-e^2x^2}{x^2\sqrt{(aa-xx)}}$. The differential of this put $=o$, and reduced gives $x^2=\frac{3a^4}{2e^2}\left\{1-\sqrt{1-\frac{8ee}{9aa}}\right\}$ this being computed we easily obtain $y=\frac{c}{a}\sqrt{(aa-xx.)}$

If the ellipsis become a circle or $e=o$, we should have $x^2=\frac{2}{3}aa$, $y^2=\frac{aa}{3}$, and the arch fallen through would be 35° 16′.

Question XVI. (123.)—*By Mr. Charles Wilder.*

Required the equation that if between it and $y^3+py^2+qy+r=0$, y be eliminated, the resulting equation will be of the form $x^6+mx^3+n=0$.

First Solution.—*By Professor Strong.*

Assume $y=\frac{x^2+m'x+n'}{x}=x+\frac{n'}{x}+m'$ for the equation sought, in which n', and m', are two quantities to be determined so as to answer the conditions. Hence I have

$$y^3=\left(x+\frac{n'}{x}\right)^3+3\left(x+\frac{n'}{x}\right)^2m'+3\left(x+\frac{n'}{x}\right)m'^2+m'^3$$

$$py^2=\left(x+\frac{n'}{x}\right)^2p+2\left(x+\frac{n'}{x}\right)m'p+m'^2p$$

$$qy=\left(x+\frac{n'}{x}\right)q+m'q$$

$$r=r$$

$$\therefore y^3+py^2+qy+r=0=x^3+\frac{n'^3}{x^3}+\left(x+\frac{n'}{x}\right)^2\times(3m'+p)+$$

$\left(x+\frac{n'}{x}\right)\times(3n'+3m'^2+2mp+q)+m'^3+m'^2p+m'q+r=0$ (by reduction). Assume $3m'+p=0$ or $m'=\frac{-p}{3}$ and $3n'=-(3m'^2+2m'p+q)=\frac{p^2}{3}-q$ (since $m'=-\frac{p}{3}$). Also $m'^3+m'^2p+m'q+r=\frac{2p^3}{27}-\frac{pq}{3}+r$. Hence our equation is reduced to $x^3+\left(\frac{\frac{p^2}{3}-q}{27.x^3}\right)^3+\frac{2p^3}{27}-\frac{pq}{3}+r=0$. Multiply by x^3, and we have $x^6+\left(\frac{2p^3}{27}\frac{-pq}{3}+r\right)x^3+\left(\frac{\frac{p^2}{3}-q}{27}\right)^3=0=x^6+mx^3+n=0$. $\left(\text{If } m=\frac{2p^3}{27}-\frac{pq}{3}+r,\ n=\frac{\left(\frac{p^2}{3}-q\right)^3}{27}=\left(\frac{p^3}{9}-\frac{q}{3}\right)^3\right)$ which equation is of the form required. The equation in x, is now under the form of a quadratic ; hence x^3, is found by the usual rules of quadratics, and thence x is had also. Then x being substituted in the equation in y, which I assumed gives y, the root of the given equation. It is evident that the result will be the same as the rule of Cardan.

QUESTION XVII. (124.)—*By* Ομικρον, *N. C.*

It is required to inscribe, in a given paraboloid, a conic frustum whose solidity shall be a maximum.

FIRST SOLUTION.—*By Mr. Eugenius Nulty.*

Let r and rx be the lower and upper bases of the frustrum, and a the altitude of the paraboloid. The height of the frustum will then be $a(1-x^2)$; and we shall have $\frac{\pi . ar^2}{3}.(1-x^2).(1+x+x^2)=$ a max. Wherefore, $1-3x^2-4x^3=0$, from which the value of x may be determined, and thence the required frustum.

Corollary. Since the preceding expression is independent of the altitude a, it is evident that the upper bases of all the greatest conic frustæ inscribed in paraboloids, having the same base, are equal to each other, and consequently, horizontal sections of a cylinder standing perpendicular to the common base of the paraboloid.

SECOND SOLUTION.—*By Mr. Henry Darnall, Phil.*

Let a= the altitude of the paraboloid, v= half the lower base, x= half the upper base of the frustum; then, $b^2 : a :: x^2 : \frac{ax^2}{b^2}$, and $a - \frac{ax^2}{b^2}$= the altitude of the frustum; therefore $\frac{3b^2}{4}$. $(b^4 + b^3x - x^4 - bx^3)$=conic frustum = a max. The fluxion of this =0, and reduced, gives $4x^3+3bx^2=b^3$; from which x will be known.

THIRD SOLUTION.—*By Mr. James Macully.*

Let a= the radius of the lower, and y= that of the upper base, and c= the parameter of the generating parabola; then by the common rule of Mensuration, the content of the frustum $\frac{\pi}{3}(a^2+ay+y^2) \cdot \frac{a^2-y^2}{c}$ = a max. its differential =0, and reduced gives $4y^3+3ay^2=a^3$, which determines y the radius of the upper base.

QUESTION XVIII. (125.)—*By Mr. Swale, Liverpool.*

Four right lines and three points are given in position; draw through one of the given points, a line meeting two of the lines given in position, so that right lines drawn from the two points of intersection to the two remaining points given in position, shall make equal angles with the two remaining lines given in position.

FIRST SOLUTION.—*By Mr. Eugenius Nulty.*

Let the equations of the required line, and the lines intersected be

$$y=tx,\ y=m(x-p),\ y=n(q-x).$$

The cordinates of the points of intersection will then be

$x=\frac{mp}{m-t}$, $y=\frac{mpt}{m-t}$, and $x=\frac{mq}{m+t}$, $y=\frac{mqt}{m+t}$, respectively, and the equations of the lines drawn from these points to the given points (α, β), (α', β') will be

$$\mu=\frac{mpt-\beta(m-t)}{mp+\alpha(m-t)}, \quad \mu'=\frac{mqt-\beta'(m+t)}{mq+\alpha'(m+t)}.$$

The angle formed by the lines corresponding to these equations is equal to that formed by the remaining given lines; wherefore we have $\frac{\mu-\mu'}{1+\mu\mu'}=n$, a given quantity, and by substitution we obtain a quadratic of the form.

$$t^2+At=B,$$

from which t may be determined by construction or calculation, and thence the portion of the required line becomes known.

QUESTION XIX. (126.)—*By Professor Strong.*

It is required to draw the shortest line possible from one given point to another on the surface of a given parabolic conoid.

FIRST SOLUTION.—*By Mr. Eugenius Nulty.*

The equation of a surface of revolution is $z=F.(x^2+y^2)=F(r^2)$, by putting $x=r\cos.\varphi$, $y=r\sin.\varphi$. Wherefore $dz=F'.dr$; and the element of a line drawn on this surface is

$$d\varsigma=\sqrt{\{r^2d\varphi^2+(1+F'^2)\,dr^2\}} \ldots\ldots (1)$$

The integral of this expression being a maximum, we have by the principles of variations $\int\delta.\{\sqrt{r^2d\varphi^2+(1+F'^2)\,dr^2}\}=0$, which expanded gives in case of $d\varsigma$ constant,

$$r^2d\varphi=cd\varsigma,$$

an equation corresponding to the shortest line that can be drawn on any surface of revolution whatever.

In the present case we have $z=\frac{r^2}{2a}$; therefore, $F'=\frac{r}{a}$; and eliminating from the equation (2) $d\varsigma$ by virtue of (1); and then $d\varphi$, we shall obtain,

$$d\varphi=\frac{cdr}{r}\cdot\sqrt{\left(\frac{a^2+r^2}{r^2-c^2}\right)}$$

$$d\varsigma = \frac{rdr}{a} \cdot \sqrt{\left(\frac{a^2+r^2}{r^2-c^2}\right)}$$

which are rationalized by assuming $\sqrt{\frac{a^2+r^2}{r^2-c^2}} = \theta$; and give

$$\varphi = c' - \left\{\text{tan.}-1\ \frac{c\theta}{a} + \log.\left(\frac{\theta-1}{\theta+1}\right)^{\frac{1}{2}}\right\},$$

$$\delta = c'' + \frac{a^2+c^2}{2a}\left\{\frac{\theta}{\theta-1} - \log.\left(\frac{\theta-1}{\theta+1}\right)^{\frac{1}{2}}\right\}.$$

The constant quantities are easily determined by the given values of r and φ, corresponding to the given points on the surface.

It has been remarked by Ομίκρον, that a general solution of this question is given in the fourth volume of John Bernoulli's works. "Lausannæ et Genevæ, sumptibus Marci Michælis Bosesquet et sociorum," 1742. It is numbered 166 and is in the following words: "In superficie quacunque curva ducere lineam inter duo punctæ brevissimam." The final equation which the author gives is "$(ds^2+dz^2)\text{T}ddy = (\text{T}dzdy - zds^2) \times ddz$" where x y and z denote three co-ordinates, $ds = (dx^2+dy^2)^{\frac{1}{2}}$ and T the subtangent expressed in terms of the coordinates.

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SECOND SOLUTION.—*By Dr. Bowditch.*

Let x be the abscis, and y the ordinate of the generating parabola whose equation is $ax = yy$, and let φ be the arch described by the parabola in its revolution to form the part of the conoid including the proposed line whose length is s; then we shall have

$$ds = \sqrt{(dx^2+dy^2+y^2d\varphi^2)}$$

whose integral gives the sought line $s = f\sqrt{(dx^2+dy^2+y^2d\varphi^2}$.

Taking now the variation of s supposing s, φ only to vary, we shall get for the equation of minimum by the usual rules of the calculation of variations $d.\frac{y^2d\varphi}{ds} = o$, whose integral is $\frac{y^2d\varphi}{ds} = c$ or $ds = \frac{y^2d\varphi}{c}$, whence we get for the proposed line

$$\frac{y^4 d\varphi^2}{c^2} = dx^2 + dy^2 + y^2 d\varphi^2$$

substituting $a dx = 2y dy$ we get $d\varphi = \frac{c}{a} \cdot \frac{dx}{x} \cdot \sqrt{\frac{(x+\xi a)}{x - \frac{cc}{a}}}$,

which becomes rational by putting $\sqrt{\frac{(x+\xi a)}{x-\frac{cc}{a}}} = z$, and we get for the integral

$$\varphi = e - \text{arc. } (\tan. \frac{2cz}{a}) + \frac{c}{a} \log. \frac{z+1}{z-1}$$

The constants c, e, are to be disposed of so as to correspond to the two proposed points of the conoid through which the line is to be drawn. Suppose them accented with one accent for the first point, and two accents for the second point, we shall have given x', y', φ', z', x'', y'', ϕ'', z'', and for these points the value of φ will become φ', φ'' and theirdifference $\phi' - \varphi''$ will give the value of c, which being substituted in ϕ' will give e.

Question. XX. (127.)—*By Mr. Eugene Nulty.*

Determine from the same expression all the small oscillations which can be made by the segment of a sphere in contact with a horizontal plane.

First Solution.—*By Dr. Adrain.*

Let each differential element dm of the solid m be referred to three rectangular axes by the co-ordinates x, y, z in small capitals, the first and second of which are in the horizontal plane on which the body moves, and the third is directed vertically upwards; also let P, Q, R be the accelerative forces acting on dm in the directions of x, y, z, and tending to increase those co-ordinates, and let dt be the constant differential of the time; then according to the general principle of Dalembert, the motion of the system will be expressed by the equation

$$(1) \int \frac{dd\mathrm{x}\delta\mathrm{x} + dd\mathrm{y}\delta\mathrm{y} + dd\mathrm{z}\delta\mathrm{z}}{dt^2} dm = \int (\mathrm{P}\delta\mathrm{x} + \mathrm{Q}\delta\mathrm{y} + \mathrm{R}\delta\mathrm{z}) dm.$$

To apply this equation to the motion of a solid, let x, y, z, be the co-ordinates of the centre of gravity of the solid referred to the formentioned axes, and let x', y', z', be the

coordinates of dm to the same axes, but having their origin in the centre of gravity of the solid ; consequently

$$x=x+x',\ y=y+y',\ z=z+z'.$$

And in the case of an uniform parallel gravity g, which is the only accelerative force, and acts in a direction tending to diminish z, we have $P=o$, $Q=o$, $R=-g$; and by substitution and reduction, Eq. (1) becomes

$$(2)\quad \int \frac{ddx'\delta x'+ddy'\delta y'+ddz'\delta z'}{dt^2}\,dm+$$
$$m\frac{ddx}{dt^2}\delta x+m\frac{ddy}{dt^2}\delta y+\left(m\frac{ddz}{dt^2}+mg\right)\delta z=0.$$

Let x'', y'', z'', be the rectangular coordinates of dm referred to the three principal axes of inertia of the solid, passing through its centre of gravity, then

$$(3)\quad \begin{aligned} x'&=\alpha x''+\beta y''+\gamma z'',\\ y'&=\alpha' x''+\beta' y''+\gamma' z'',\\ z'&=\alpha'' x''+\beta'' y''+\gamma'' z'' ; \end{aligned}$$

the nine coefficients α, β, α', &c. being given functions of three independent quantities $\downarrow$, φ, θ, as is shewn in the Mecanique Celeste, vol. i, page 73.

Now the co-ordinates x'', y'', z'', of dm remain invariable, while α, β, α', &c. vary by the motion of the body ; we may therefore find the values of ddx', ddy', ddz', $\delta x'$, $\delta y'$, $\delta z'$, by means of eq. (3), and putting $a=\int x''^2\,dm$, $b=\int y''^2\,dm$, $c=\int z''^2\,dm$, the part of eq. 2 having the integral sign. $\int$ is tranformed into the formula

$$(4)\quad \frac{1}{dt^2}\cdot\left\{\begin{aligned} &a(dd\alpha\delta\alpha+dd\alpha'\delta\alpha'+dd\alpha''\delta\alpha'')\\ &+b(dd\beta\delta\beta+dd\beta'\delta\beta'+dd\beta''\delta\beta'')\\ &+c(dd\gamma\delta\gamma+dd\gamma'\delta\gamma'+dd\gamma''d\gamma'') \end{aligned}\right\}$$

That we may express this formula in the simplest manner, let us put $a+b=c$, $a+c=B$, $b+c=A$, and make the following substitutions,

$$(5)\quad \begin{aligned} d\varphi-d\downarrow\cos.\theta&=pdt,\\ d\downarrow\sin.\theta\sin.\phi-d\theta\cos.\varphi&=qdt,\\ d\downarrow\sin.\theta\cos.\phi-d\theta\sin.\phi&=rdt ; \end{aligned}$$

in which p, q, r, denote the angular velocities of the solid about the three axes of z'', x'', y'', as in the notation of La Place.

We have now to compute the values of dda, dda', &c. δa, $\delta a'$, &c. in terms of $\downarrow$, ϕ, θ ; and the formula (4) will be transformed into

(6) $\quad \text{M}\delta\downarrow+\text{M}'\delta\varphi+\text{M}''\delta\theta$;

the values of M, M′, M″ being as follows

$$\text{M}dt=d\{\text{A}q\sin.\theta\sin.\phi+\text{B}r\sin.\theta\cos.\phi-cp\cos.\varphi\},$$

$$\text{M}'dt=d(cp)-\text{A}q(d\downarrow\sin.\theta\cos.\varphi+d\theta\sin.\varphi)+\text{B}r(d\downarrow$$

(7) $\sin.\theta\sin.\varphi-d\theta\cos.\phi),$

$$\text{M}''dt=d(\text{B}r\sin.\phi-\text{A}q\cos.\phi)-d\downarrow(\text{A}q\cos.\theta\sin.\phi$$

$$+\text{B}r\cos.\theta\cos.\phi+cp\sin.\theta).$$

Thus eq. (2) is reduced to

$$(8)\ m\frac{ddx}{dt^2}\delta x+m\frac{ddy}{dt^2}\delta y+(m\frac{ddz}{dt^2}+mg)\delta z+\text{M}\delta\downarrow+\text{M}'\delta\phi+\text{M}''$$

$$\delta\theta=o\ ;$$

which is the general equation of the motion of any solid whatever when subjected to the action of an uniform and parallel gravity.

In the present question the surface of the solid is in contact with the plane of x, y ; therefore if R= radius of the sphere, a= the distance from the centre of gravity of the solid to the centre of the sphere, we have the equation following which denotes the contact

(9) $\quad z+a\cos.\theta=\text{R}$;

therefore (10) $\quad \delta z-a\delta\theta\sin.\theta=0.$

By means of eq. (10) we may eliminate δz or $\delta\theta$ from eq. (8) : or we may multiply eq. (10). See the indeterminate λ, and add the result to equ. (8) which will thus become

$$(11)\quad m\frac{ddx}{dt^2}\delta x+m\frac{ddy}{dt^2}\delta y+(m\frac{ddz}{dt^2}+mg+\lambda)\delta z$$

$$+\text{M}\delta\downarrow+\text{M}'\delta\phi+(\text{M}''-\lambda a\sin.\theta)\delta\theta=0.$$

which is the general equation of the motion of the segment when in contact with a horizontal plane.

If the motion be subject to any particular conditions, these may be expressed by equations among the variables $x, y, z, \downarrow, \phi, \theta$, the variations of which equations multiplied by indeterminate coefficients will give one or more expressions of the form $v\delta x+v'\delta y+$ &c. which being added to eq. (11) will furnish the general equation of motion, comprehending all such particular cases.

When there are no particular conditions the six variations of $x, y, z, \downarrow, \varphi, \theta$ in eq. (11) may be considered as arbitrary on account of the indeterminate λ, and therefore the six coefficients of these variations must be each separately equal to zero ; we have therefore the six equations

$$(12)\qquad m\frac{ddx}{dt^2}=o,\ m\frac{ddy}{dt^2}=o,\ m\frac{ddz}{dt^2}+mg+\lambda=o,$$
$$\text{M}=o,\ \text{M}'=o,\ \text{M}''-\lambda a\sin.\theta=o.$$

The first and second of these equations give $x=\text{c}'t+\text{D}'$, $y=\text{c}''t+\text{D}''$, which shew that the centre of gravity of the solid continues in one vertical plane, and that its horizontal motion is uniform.

The three equations in M, M′, M″, of eq. (12) being reduced so as to separate p, q, r from ψ, ϕ, θ, with the third of eq. (12) in $\frac{ddz}{dt^2}$, and the equation of contact give the five following,

$$\text{C}\frac{dp}{dt}+\frac{\text{B}-\text{A}}{\text{AB}}qr=0.$$
$$\text{A}\frac{dq}{dt}+\frac{\text{C}-\text{B}}{\text{BC}}.pr-\lambda a\sin.\theta\cos.\phi=0,$$
$$(13)\qquad \text{B}\frac{dr}{dt}+\frac{\text{A}-\text{C}}{\text{AC}}.pq+\lambda a\sin.\theta\sin.\varphi=0,$$
$$m\frac{ddz}{dt^2}+mg+\lambda=0,$$
$$z+a\cos.\theta-\text{R}=0;$$

from which and eq. (5) we may deduce the values of ψ, φ, θ, z, p, q, r; and these with the values already obtained of x and y determine the position and motion of the segment at every instant, whether the motions be finite or infinitely small.

When the oscillations are indefinitely small, θ, q, r, $\frac{ddz}{dt^2}$ are all indefinitely small, in which case the preceding equations are susceptible of complete integration.

The product qr in the first of equ. (13) being of the second degree may be omitted; whence $dp=o$ or p is constant, which shews that the rotation about the diameter of the sphere passing through the centre of gravity is uniform, which also results from the known equation $\text{B}-\text{A}=o$.

Again because $\frac{ddz}{dt^2}$ is indefinitely less than g, the fourth of eq. (13) gives $mg+\lambda=o$ or $\lambda=-mg$, whence putting $\frac{\text{C}-\text{A}}{\text{A}}p=h$, $\frac{amg}{\text{A}}=k$, and sin. $\theta=\theta$, the 2d and 3d of eq. (13) become

$$(14)\qquad \begin{aligned}\frac{dq}{dt}&=-hr+k\theta\cos.\phi,\\ \frac{dr}{dt}&=hq-k\theta\sin.\phi.\end{aligned}$$

Also eq. (5) are in this case reducible to

$$(15)\qquad \begin{aligned}&d\phi-d\downarrow=pdt,\\ &d\downarrow\theta\sin.\varphi-d\theta\cos.\varphi=qdt,\\ &d\downarrow\theta\cos.\varphi+d\theta\sin.\varphi=rdt\end{aligned}$$

The first of these gives $d\downarrow=d\phi-pdt$, and

$$(16)\qquad \downarrow=\phi-pt+\kappa.$$

And putting for $d\downarrow$ its value in the 2d and 3d of eq. (15) they become

$$(17)\qquad \begin{aligned}&d(\theta\cos.\varphi)+pdt(\theta\sin\varphi)=-qdt,\\ &d(\theta\sin.\varphi)-pdt(\theta\cos.\varphi)=rdt.\end{aligned}$$

Now put $\theta\sin.\varphi=s$, $\theta\cos.\varphi=u$, and eq. (17) with the 2d and 3d of 15 become

$$(18)\qquad \begin{aligned}\frac{dq}{dt}&=-hr+ku,\\ \frac{dr}{dt}&=hq-ks,\\ q&=-\frac{du}{dt}-ps,\\ r&=\frac{ds}{dt}-pu.\end{aligned}$$

Eliminate q and r from eq. (18) which is easily done by differentiation and substitution ; and putting $h-p=b$, $hp+k=c$, eq. (18) all reduced to the two following

$$(19)\qquad \begin{aligned}&\frac{dds}{dt^2}+b\frac{du}{dt}+cs=0,\\ &\frac{ddu}{dt^2}-b\frac{ds}{dt}+cu=0.\end{aligned}$$

Eliminate $\frac{ddu}{dt^2}$ from eq. (19) by differentiation and subtraction, and

$$(20)\qquad \frac{d^3s}{dt^3}+(b^2+c)\frac{ds}{dt}-bcu=0.$$

In a similar manner eliminate u and $\frac{du}{dt}$ from eq. (20) and the first of (19), and we have

$$(21)\qquad \frac{d^4s}{dt^4}+(b^2+2c)\frac{d^2s}{dt^2}+c^2s=0.$$

This eq. (21) may be integrated by any of the known methods for linear equations, and u becomes known by eq. (20).

Thus we obtain the following values of s and u,

$$(22)\qquad \begin{aligned} s&=\text{E}\sin.(nt+\varepsilon)+\text{E}'\sin.(n't+\varepsilon'),\\ u&=\text{E}\cos.(nt+\varepsilon)-\text{E}'\cos.(n't+\varepsilon'); \end{aligned}$$

in which E, E′, ε, ε', are arbitrary constants, and n, n' are the two positive roots of the equation

$$n^4-(b^2+2c)n^2+c^2=0,$$

or which is the same thing $(n^2+bn-c).(n^2-bn-c)=0$; so that $n=-\frac{b}{2}+\frac{1}{2}\sqrt{(b^2+4c)}$, and $n'=\frac{b}{2}+\frac{1}{2}\sqrt{(b^2+4c)}$.

Now s and u being known, φ is known by the eq. tan. $\varphi=\frac{s}{u}$, and ψ is known by eq. (16) ; also q and r are determined by the 3d and 4th of eq. (18).

The value of θ is derived from the equation $\theta^2=s^2+u^2$, which by substituting for s and u their values in eq. (22) gives

$$(23)\qquad \theta^2=2(\text{E}^2+\text{E}'^2)+2\text{E}\text{E}'.\cos.(\mu^{\frac{1}{2}}t+\varepsilon+\varepsilon'),$$

in which $\sqrt{(\mu)}=n+n'=\sqrt{(b^2+4c)}$; and z is determined from the value thus found of θ^2 by the equation

$$z+a\cos.\theta=\text{R}.$$

Lastly, the time T of oscillation in which θ and z change from the greatest to the least values is found from eq. (23)

$$\text{T}=\frac{\pi}{\sqrt{(\mu)}}=\frac{\pi\text{A}}{(\text{C}^2p^2+4amg\text{A})^{\frac{1}{2}}}$$

Second Solution.—*By Mr. Eugenius Nulty.*

Conceive a horizontal axis to pass through the centre of the spheric segment at right angles to the plane in which the oscillations take place ; and at the end of the time t, let r be the distance of this axis from a vertical plane parallel to it, and given in position ; ρ and ρ' the distances of a particle dm, and the centre of gravity of the segment from the same axis, φ the angle which ρ' forms with a vertical passing through its extremity, a the radius of the segment, g the force of gravity, and F the friction of the horizontal plane.

The forces urging the particle dm during the instant

dt are evidently $\frac{d^2r}{dt^2}$ in the horizontal direction r, $\rho\frac{d^2\varphi}{dt^2}$ in the direction of $\rho\varphi$, and the centrifugal force $\rho\frac{d\varphi^2}{dt^2}$ in the direction of ρ. The moments of these forces in the direction r and ϕ, and extended to all the particles of the body m are easily seen to be

$$\left(\frac{d^2r}{dt^2}.m+\frac{d^2\varphi}{dt^2}.m\rho'\cos.\varphi-\frac{d\varphi^2}{dt^2}.m\rho'\sin.\varphi\right)\delta r,$$

$$\frac{d^2r}{dt^2}.m\rho'\cos\varphi+\frac{d^2\phi}{dt^2}.\int\rho^2dm-mg\rho'\sin.\phi).\ \delta\varphi,$$

and the moments of the friction in the same derections are $\mathrm{F}\delta r$, $\mathrm{F}\delta a\varphi$. We have therefore

$$\left(\frac{d^2r}{dt^2}.m+\frac{d^2\phi}{dt^2}.m\rho'\cos.\phi-\frac{d\varphi^2}{dt^2}.m\rho'\sin.\varphi+\mathrm{F}\right)\delta r$$

$$+\left(\frac{d^2r}{dt^2}.m\rho'\cos.\phi+\frac{d^2\varphi}{dt^2}.m\int\rho^2dm-mg\rho'\ \sin.\ \varphi+a\mathrm{F}\right)$$

$$\delta\varphi=o,$$

an equation which includes all the oscillations of the segment.

If we consider $\mathrm{F}=o$, and consequently δr and $\delta\phi$ independent; and also neglect $\frac{d\varphi^2}{dt^2}$, we shall have in case of sliding motion and small oscillations

$$\frac{d^2\phi}{dt^2}.(\int\rho^2dm-\rho'^2m)=mg\rho'\phi. \qquad (1)$$

If we assume $a\delta\phi=-\delta r$, and consequently $ad\phi=-dr$. we shall have in case of small oscillations and rolling motion.

$$\frac{d^2\phi}{dt^2}.(\int\rho^2dm+a^2m-2a\rho'm)=mg\rho'\phi, \qquad (2)$$

which give for the time of oscillation

$t=\pi.\frac{k}{\sqrt{(g\rho')}}$ and $t=\pi.\sqrt{\left\{\frac{(a-\rho')^2+k^2}{g\rho'}\right\}}$, in which ρ'^2+k^2 is put for $\int\frac{\rho^2dm}{m}$.

It may not be improper to remark that the equations here found are not confined to the oscillations of the segment of a sphere. They will determine the oscilla-

tions of an indefinite number of other solids, as the investigation requires that the portion $a\phi$ of a section of the body at the point of contact and in the plane of oscillation should be a circular arc, and that the body be symmetrical with respect to the plane in which it oscillates.

THIRD PRIZE SOLUTION.—*By Dr. Anderson.*

I am not certain whether the proposer of this question refers to a homogeneous or a heterogeneous segment, or whether other forces than gravity and the reaction of the plane are intended to be included or not. The subjoined solution, however, extends to all bodies whatever, of every figure and law of density, moved by any initial impulses compatible with small oscillations, with a particular examination of the case in which the body rocks and spins about its vertical, but does not slide.

Let x, y, z, be the rectangular co-ordinates of any element Dm of the body referred to planes fixed in space the horizontal being that of xy; x_1, y_1, z_1, co-ordinates of Dm originating at the centre of gravity and fixed in the body, z_1 being vertical when the body is in equilibrium; X, Y, Z, x', y', z', the co-ordinates (referred to the first axes) of the centre of gravity, and of the centre of curvature at the lowest point of the body at rest; e the distance between these centres; A, B, C, p, q, r, the moments of inertia and velocities of rotation round the body-axes; F, G, H, P, Q, R, the integrals Sy_1z_1Dm, Sz_1x_1Dm, Sx_1y_1Dm, $\int pdt$, $\int qdt$, $\int rdt$; a, b, c, a' &c. the cosines of the angles xx_1, xy_1, &c; g the force of gravity, and M the body's mass. We have in the first place (1)

$$\begin{aligned} x &= X + ax_1 + by_1 + cz_1, \\ y &= Y + a'x_1 + b'y_1 + c'z_1, \\ z &= Z + a''x_1 + b''y_1 + c''z_1, \\ x' &= X + ce, \\ y' &= Y + c'e \\ z' &= Z + c''e \end{aligned}$$

The general formula of dynamics applicable to this case is

$$SmD\left\{\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \left(\frac{d^2z}{dt^2} + g\right)\,\delta z\right\} = 0$$

Substituting the above values of x, y, z, and reducing by means of the formulæ

$$\begin{aligned}\delta a &= b\delta R - c\delta Q\\ \delta a' &= b'\delta R - c'\delta Q\\ \delta a'' &= b''\delta R - c''dQ, \&c.\end{aligned}$$

$$\begin{aligned}a d^2a + a'd^2a' + a''d^2a'' &= -(q^2+r^2)\,dt^2\\ b d^2b + b'd^2b' + b''d^2b'' &= -(r^2+p^2)\,dt^2\\ c d^2c + c'd^2c' + c''d^2c'' &= -(p^2+q^2)\,dt^2, \&c,\end{aligned}$$

we obtain the transformed equation

$$\frac{d^2X}{dt^2}\,\delta X + \frac{d^2Y}{dt^2}\,\delta Y + \frac{1}{M}\,(P'\delta P + Q'\delta Q + R'\delta P) = 0$$

where P', Q', and R' will be found to be the same with the left members of the three equations (Méc. Anal. Vol. II. p. 266) investigated by another method. Lastly, substituting the values of δX and δY, there results (2)

$$\frac{d^2X}{dt^2}\,\delta x' + \frac{d^2Y}{dt^2}\,\delta y' + P''\delta P + Q''\delta Q + R''\delta R = 0$$

a general expression from which all the oscillations of which the body is susceptible may be fully determined.

First. If there be nothing to prevent the body's sliding, then the above variations are independent of each other, and we have, (beside $Z = z' - c''e$),

$$\frac{d^2X}{dt^2} = 0,\quad \frac{d^2Y}{dt^2} = 0,\quad P' = 0,\quad Q' = 0,\quad R' = 0.$$

Hence the projection of the centre of gravity will move uniformly forward in a straight line, at the same time that the body will oscillate about its centre of gravity, precisely as it would if it were suspended by a fixed point at the centre of lowest curvature, with its moments of inertia A and B diminished by Me^2; C, F, G, H remaining as before. As a complete solution to this latter problem is given with all possible generality by Lagrange (Méc. Anal. Vol. II. p. 263 to p. 276.), I shall merely add that in the case of a homogeneous segment, and in the numerous other cases where $A = B$, $F = 0$, $G = 0$, $H = 0$, the formulæ are much simplified and we obtain

$$\begin{aligned}a'' &= \alpha \sin.\ \mu + \alpha' \sin.\ \mu'\\ b'' &= \alpha \cos.\ \mu + \alpha' \cos.\ \mu'\\ p &= \varepsilon \sin.\ \mu + \varepsilon' \sin.\ \mu'\\ q &= \varepsilon \cos.\ \mu + \varepsilon' \cos.\ \mu'\end{aligned}$$

where $\mu = \rho nt + \beta$, $\mu' = \rho' nt + \beta'$, n being the velocity of

rotation (invariable in the present case) round the axis of z, ; α, α', β', β', arbitrary constants, and ρ, ρ', ε, ε', functions of the arbitrary and other constant quantities. From the above equations, the position of the sphere may be readily determined for any required epoch. It will easily appear for instance, that if two unequal radii α and α' be supposed to revolve about the same fixed point with uniform but different velocities in opposite directions, the line which joins their extremities will at all times represent the obliquity of the segment's base ; while the longitude of the line of equinoxes, and the distance of a fixed meridian of the sphere from the same line, will be represented by the angles which the first mentioned line makes with two others passing through the fixed point, one stationary and the other revolving uniformly with the velocity of the rotation of the sphere.

Secondly. Suppose it be required to find the small oscillations of any given body whatever, spinning and rocking freely but not sliding, on a horizontal plane. It is clear that in this case, the velocity of the point of contact in the direction of the co-ordinates x and $-y$ will be equal to the velocities of rotation round the axes y and x multiplied by h, the radius of lowest curvature. We shall therefore, by known formulæ, have (3)

$$\delta x' = h\,(a'\delta P + b'\delta Q + c'\delta R)$$
$$\delta y' = -h\,(a\,\delta P + b\,\delta Q + c\,\delta R)$$

Substituting these values in eq. (2) there will result an equation of this form

$$P_{,}\delta P + Q_{,}\delta Q + R_{,}\delta R = 0\,;$$

whence we obtain the three equations

$$P_{,} = 0, \quad Q_{,} = 0, \quad R_{,} = 0.$$

After the requisite substitutions and reductions, $P_{,}$, $Q_{,}$, and $R_{,}$ will be transformed to functions of a'', b'', r and their differentials, and results will be obtained in all respects identical with those exhibited below, but which I have preferred to investigate by Lagrange's method as being more commodious in the present instance. Denoting by T'' half the sum of all the living forces of the system, by T' that part of T'' which is due to the progressive motion of the centre of gravity, and by T that which is due to the motion of rotation round that centre, we shall have

$$T+T' = T''.$$

With regard to T, it retains the same form as at p. 264. Vol. II. Méc. Anal., recollecting only that as the integrals A, B, C, F, G, H, are to be referred to the centre of gravity and not to the point of suspension, A and B must each be diminished by Me^2. As respects T', its value, which was constant in the first part of this problem, will now be variable and may be calculated thus. By means of equations (1), the square of the velocity of the centre of gravity is transformed first to

$$dx'^2+dy'^2-2e\,(dcdx'+dc'dy')+e^2\,(dp^2+dq^2)$$

then by means of eq. (3) and other known formulæ, to

$$\begin{aligned} &h^2\left\{p^2+q+r^2-(a''p+b''q+c''r)^2\right\}\\ -&2eh\left\{(p^2+q^2)c''-(a''p+b''q)\,r\right\}\\ +\;&e^2\left\{p^2+q^2\right\}; \end{aligned}$$

and finally in the case of small oscillations (the above formulæ not being confined to these), taking care to omit no terms but such as must disappear after the variation of T, into

$$(kp-ha''r)^2+(kq-hb''r)^2,$$

where $k=h-e$. As T' is therefore a function not only of p, q and r, but also of a'' and b'' we shall have

$$\delta T'=\frac{dT'}{dp}\,\delta p+\frac{dT'}{dq}\,\delta q+\frac{dT'}{dr}\,\delta r+\frac{dT'}{da''}\,\delta a''+\frac{dT'}{db''}\,\delta b'';$$

whence $\left\{d\frac{\delta T'}{\delta dP}-\frac{\delta T'}{\delta P}\right\}\delta P+\&c.$ becomes

$$\begin{aligned} &\left\{d.\,\frac{dT'}{dp}\,\frac{1}{dt}+q\,\frac{dT'}{dr}-r\,\frac{dT'}{dq}-\frac{dT'}{db''}\right\}\delta P\\ +&\left\{d.\,\frac{dT'}{dq}\,\frac{1}{dt}+r\,\frac{dT'}{dp}-p\,\frac{dT'}{dr}+\frac{dT'}{da''}\right\}\delta Q\\ +&\left\{d.\,\frac{dT'}{dr}\,\frac{1}{dt}+p\,\frac{dT'}{dq}-q\,\frac{dT'}{dp}-b''\,\frac{dT'}{da''}+a''\,\frac{dT'}{db''}\right\}\delta R \end{aligned}$$

or, substituting the values of the partial differentials,

$$\begin{aligned} &M\left\{k\,\frac{d(kp-ha''r)}{dt}+er\,(kq-hb''r)\right\}\delta P\\ +&M\left\{k\,\frac{d(kq-hb''r)}{dt}-er\,(kp-ha''r)\right\}\delta Q, \end{aligned}\qquad (4)$$

It would be easy to show in this case as Lagrange has

done in the case of the revolving pendulum, that the centrifugal force will swing the upright axis of the body to a finite distance from the vertical unless either r be very small or F and G be very small. In the first case, the preceding expression reduces itself to

$$Mk^2 \left\{ \frac{dp}{dt} . \delta P + \frac{dq}{dt} . \delta Q \right\}$$

and as the values of T and V remain as before, it follows the small oscillations of a body of any form and law of density, rocking on a horizontal plane, spinning slowly and not sliding, will be the same precisely as if the body were suspended freely like a revolving pendulum about a fixed point at the centre of lowest curvature, the moments of inertia A and B being diminished at the same time each by $M(e^2 - k^2)$. It appears also that these oscillations are the same as if there were no friction, but the axis of z loaded with two masses each equal to $\frac{1}{2} M$ at two points distant k on each side of the centre of gravity.

Lastly let F and G be supposed to be very small, the spinning motion having any degree of velocity. This velocity will now remain invariable, and formula (4) becomes, substituting for pdt and qdt their values $na''dt + db''$, $nb''dt - da''$,

$$M \left\{ k^2 \frac{d^2 b''}{dt^2} - 2\, ken \frac{da''}{dt} - (e^2 n^2 - ge) \right\} \delta P$$
$$- M \left\{ k^2 \frac{d^2 a''}{dt^2} - 2\, ken \frac{db''}{dt} - (e^2 n^2 - ge) \right\} \delta Q.$$

These coefficients are to be added to the coefficients of δP and δQ in the case of the revolving pendulum, and the complete solution of this part of the problem will then be found, by the ordinary methods, to be

$$a'' = a + \alpha \sin. \eta t + \alpha' \sin. \eta' t + \alpha'' \cos. \eta t + \alpha''' \cos. \eta' t,$$
$$b'' = b + \beta \sin. \eta t + \beta' \sin. \eta' t + \beta'' \cos. \eta t + \beta''' \cos. \eta' t;$$

the constants, of which four are arbitrary, being the same with those which belong to the two equations (Méc. Anal. Vol. II. p. 271) taking care to add in the result Mk^2 to A and B, $2Mkh$ to C, and $-Mh^2$ to L.

QUESTION XXI. (128.)—*By a Correspondent.*

Suppose a heavy body, as a point, be suspended from a fixed point by an inflexible thread void of gravity, and pro-

jected in a direction inclined to the vertical plane passing through the thread, and thus made to revolve in a spherical surface whose centre is the point of suspension. It is required to ascertain the best method of finding the situation of the body at any time, and to determine particularly the times and places corresponding to its greatest ascents and descents, and generally the horizontal angle described about the vertical which passes through the point of suspension.

—

FIRST SOLUTION.—*By Dr. Bowditch.*

This question has been treated of by several authors, particularly by La Place in his *Mec.* Cel. vol. 1, page 28—30, in which the integrals are found by means of series, and the method of series is also used in Mr. Nulty's solution published in the last volume of the Transactions of the Philosophical Society of Philadelphia. The object of the proposer of this question was to shew the use of Le Gendre's "*Tables Ellyptiques*," published in the third volume of his work "*Exercises de Calcul Integral*," particularly as it regards the horizontal angle described about the vertical, which is reduced to a finite form, in computing the angle corresponding to a whole vibration, instead of the very complicated series usually given. It will not therefore be necessary to go through the preliminary calculations of the formulas, it will suffice to take them as they are given in La Place's work; but in order not to interfere with Le Gendre's notation, I shall change La Place's symbols, b, c', λ, θ, into a', ϵ, c, φ respectively; and it may not be amiss to mention that Le Gendre uses the following abridged symbols $b = \sqrt{(1-cc)}$, $\Delta(c\varphi) = \sqrt{(1-c^2 \sin.\ \varphi^2)}$, $\Delta(b\varphi) = \sqrt{(1-b^2 \sin.\ \varphi^2)}$; $\mathrm{F}(c\varphi) = \int \frac{d\varphi}{\Delta(c\varphi)}$; $\mathrm{E}(c\varphi) = \int d\varphi . \Delta(c\varphi)$ $\Pi\ (nc\varphi) = \int \frac{d\varphi}{(1+n \sin.\ \varphi^2)} \Delta.(c\varphi)$, these functions F, E, Π being what he calls elliptical functions of the *first*, *second* and *third* species. When φ becomes 90°, these expressions are denoted by $\mathrm{F}'(c)$, $\mathrm{E}'(c)$ $\Pi'(nc)$ respectively. This being premised, we shall now proceed to give La Place's solution.

The point of suspension of the pendulum (or the cen-

tre of the spherical surface) is taken for the origin of the rectangular co-ordinates x, y, z, z being vertical, and $r=\sqrt{(x^2+y^2+z^2)}$ being the length of the pendulum or the radius of that surface. a is the greatest value of z, a' the least value of z (both being considered as positive), $2g=$ the velocity acquired in falling freely by gravity in one second of time. Moreover for brevity the following abbreviations are used

$$c^2=\frac{a^2-a'^2}{r^2+a^2+2aa'};\ \text{sin. } \varphi=\sqrt{\frac{(a-z)}{a-a'}},\ \text{or, } z=a-(a-a')\times \text{sin. } \varphi^2,\ h^2=\frac{2r^2.(a+a')}{g(r^2+a^2+2aa')},\ \text{tang. } \varpi=\frac{y}{x},\ \xi^2=\frac{2g(r^2-a^2).(r^2-a'^2)}{a+a'},\ n=-\frac{(a-a')}{r+a},\ n'=\frac{a-a'}{r-a}.$$

Then by page 30 of La Place's work we have $dt=\frac{hd\varphi}{\Delta(c\varphi)}$ and $d\varpi=\frac{\xi dt}{r^2-z^2}$. The integral of the first expression gives at once $t=h.\text{F}(c\varphi)$, depending on an elliptical function of the *first species*, and if t becomes (t) when $\varphi=90°$, we shall have $t=h\text{F}'(c)$. This last integral is computed for every value of c in Table I of Le Gendre, and the former $\text{F}(c\varphi)$ is computed in Table IX for every value of c, φ, observing that in these tables instead of c he uses θ, making sin. $\theta=c$. Nothing can be more easy than this process, it is like finding the logarithm of a number from the tables, instead of going through the labour of computing it, which former methods would require.*

THIRD SOLUTION.—*By Dr. Adrain.*

Let $r=$ the length of the thread or pendulum, $a=$ the greatest and $b=$ the least distance of the body from the horizontal plane passing through the point of suspension $z=$ the distance of the body from the same plane, these distances being reckoned downwards, sin $\theta=\sqrt{\frac{a-z}{a-b}}$, g = gravity,

$$\gamma=\frac{a^2-b^2}{(a+b)^2+r^2-b^2},\ M=\frac{r\sqrt{2(a+b)}}{\sqrt{g((a+b)^2+r^2-b^2)}}.$$

By the method of La Place or otherwise, we obtain the equation

* *Errata.* In the above solution, for $\int\frac{d\varphi}{(1+n\text{ sin. }\varphi^2}\Delta.(c\varphi)$, read $\int\frac{d\varphi}{(1+n\text{ sin. }\varphi}\frac{}{)\Delta.(c\varphi)}$.

$$dt = \frac{M d\theta}{\sqrt{(1-\gamma^2 \sin.^2\theta)}}.$$

Instead of integrating this formula for the value of dt by reducing the surd denominator to a series which is the method of La Place we may proceed in the following manner.

Because $\sin.^2\theta = \frac{1-\cos.\ 2\theta}{2}$ therefore by substitution we have

$$1-\gamma^2 \sin.^2\theta = 1-\frac{\gamma^2}{2}+\frac{\gamma^2}{2}\cos.\ 2\theta = (1-\frac{\gamma^2}{2})\times(1+\frac{\frac{\gamma^2}{2}}{1-\frac{1}{2}\gamma^2}\cos.\ 2\theta).$$

and if $\frac{\frac{1}{2}\gamma^2}{1-\frac{1}{2}\gamma^2} = e$, $2\theta = \varphi$ the equation in t and θ becomes

$$dt = \frac{M}{2\sqrt{(1-\frac{1}{2}\gamma^2)}} \frac{d\phi}{\sqrt{(1+e\cos.\varphi)}}.$$

By the binomal theorem we have

$$dt = \frac{M}{2\sqrt{(1-\frac{1}{2}\gamma^2)}} d\phi \left\{ 1-\frac{1.3}{2.4} e\cos.\varphi+\frac{1.3.5}{2.4.6} e^2\cos.^2\phi-\frac{1.3.5.7}{2.4.6.8} e^3\cos.^3\varphi+\&c. \right\},$$

which integrated from $z=a$ to $z=b$, that is from $\theta=o$ to $\theta=\frac{\pi}{2}$, or which is the same thing from $\phi=o$ to $\varphi=\pi$, and the result doubled to obtain the time T of a whole vibration from $z=a$, to $z=a$, we have

$$T = \frac{M\pi}{\gamma(1-\frac{1}{2}\gamma^2)}. \left\{ 1+\frac{1.3}{4.4}e^2+\frac{1.3.5.7}{4.4.8.8}e^4+\frac{1.3.5.7.9.11}{4.4.8.8.12.12}e^6+\&c. \right\}.$$

When the motion is in a vertical plane $a=r$, and the time of vibration becomes

$$T = \frac{2r\pi}{\sqrt{(3r+b)g}}. \left\{ 1+\frac{1.3}{4.4}\left(\frac{r-b}{3r+b}\right)^2+\frac{1.3.5.7}{4.4.8.8}\left(\frac{r-b}{3r+b}\right)^4+\&c. \right\}.$$

This series for determining the time of vibration of a given pendulum converges with much greater rapidity

than that given by La Place and other writers on the motion of pendulums.

When the pendulum oscillates through a semicircle $b=c$, and the equation for the time of vibration becomes

$$T=\frac{\pi}{\sqrt{3}}\sqrt{\frac{r}{g}}\cdot\left\{1+\frac{1.3}{4.4}\cdot\frac{1}{9}+\frac{1.3.5.7}{4.4.8.8}\cdot\frac{1}{9.9}+\&c.\right\}.$$

Summing up seven terms of this series we have

$$T=\pi\sqrt{\frac{r}{g}}\times(1.1803406).$$

When the vibration is performed in a small arc, b is nearly equal to r, and we have a very near approximation by retaining only the first term of the series which gives

$$T=\frac{2r\pi}{\sqrt{(3r+b)g}}.$$

This method may also be applied to the determination of the horizontal angle described by the body in the time of one oscillation.

QUESTION XXIII. (130.) **OR PRIZE QUESTION.**

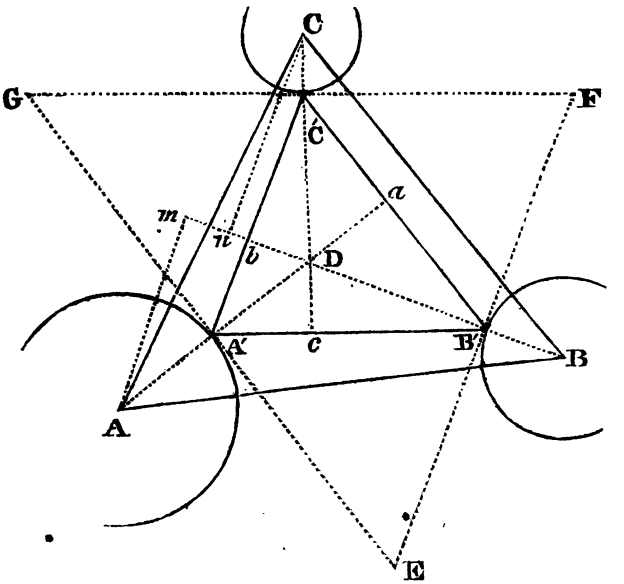
By Mr. John Smith, Cincinnati, State of Ohio.

It is required to determine the greatest or least triangle, having its angular points on the circumferences of three circles in the same plane not meeting each other, the three circles being given both in magnitude and position.

FIRST PRIZE SOLUTION.—*By Dr. Bowditch.*

Let A, B, C be the centres of the given circles A′B′C′ the minimum triangle in which the points A′, B′, C′, are situated on the circumferences of the proposed circles. Then a slight attention to the nature of the minimum will show that the tangent EA′G of the circle passing through A′ ought to be parallel to the opposite base B′C′ and in like manner the tangent EB′F is parallel to A′C′ and the tangent FC′G parallel to A′B′. In other words if the radius AA′ be continued to meet the opposite side B′C in a, it will form a right angle at a; the line BB′ continued to b will be perpendicular to A′C′: and CC′ continued to c will be perpendicular to A′B′. These three perpendiculars intersect each other in a point D. It may also be observed that the three tangents above drawn form a triangle EFG similar to A′B′C′ and of double the linear dimensions so that C′G=C′F=A′B′; B′F=B′E=A′C′; A′G=A′E

$=B'C'$, and that this triangle also possesses the property of being the least triangle that can be formed by tangents drawn to the proposed circles. So that it is the same thing to draw a *minimum* triangle whose *angula points* are situated in the circumferences of the same circles.



What has been said relative to the minimum triangle applies with a slight modification to the maximum triangle. The only difference consists in taking the angular points on the outer parts of the circles For here (as in the case of the minimum triangle) any side of the maximum triangle will be parallel to the tangents of the circle drawn from the opposite angular point; and the triangle formed by these tangents will be the maximum triangle circumscribing and touching these circles. We shall confine hereafter our remarks to the inscribed or minimum triangle, the remark made on it may be applied to the maximum triangle with the proper changes.

Let the angles $CAB=2A_1$, $ABC=2A_2$, $ACB=2A_3$, $A'AB=A_1-x_1$, $A'AC=A_1+x_1$, $B'BA=A_2+x_2$, $B'BC=A_2-x_2$; $C'CB=A_3+x_3$; $C'CA=A_3-x_3$. Upon Bb continued let fall

the perpendiculars Am, cn, then we have $A_1+A_2+A_3=90°$, $ADm=90°-A_3+x_2-x_1$; $CBm=90°-A_1+x_3-x_2$. Then $Bm=AB.\cos.ABm$, $Bn=BC.\cos.CBm$, $bm=AA'.\cos.ADm$, $bn=cc'.\cos.CDm$, hence if we put $AA=r_1$, $BB'=r_2$, $cc'=r_3$ and substitute the above symbols in $bm-bn=Bm-Bn$ we get by reduction and putting for brevity

$$b_1=\frac{(a_3-a_2).\cos.A_2}{\sin.(A_3-A^2)}\ ;\ b_2=\frac{(a_1-a_3).\cos.A3}{\sin.(A_1-A_3)}\ ;\ b_3=\frac{(a^2-a_1)\cos._1}{\sin.(A_2-A_1)}$$

the second of the three following equations, and by a mere change of the accents we get the two other equations corresponding to perpendiculars let fall on Aa, cc, continued. This system of equations after proper reductions becomes of the following form

$b_1.\sin.(A_3-A_2+x_1)=r_2.\sin.(A_3+x_1-x_2)-r_3.\sin.(A_2+x_3-x_1)$

$b_2.\sin.(A_1-A_3+x_2)=r_1.\sin.(A_1+x_2-x_3)-r_1.\sin.(A_3+x_1-x_2)$

$b_3.\sin.(A_2-A_1+x_3)=r1.\sin.(A_2+x_3-x_1)-r_2.\sin.(A_1+x_2-x_3)$.

which are the three equations to find x_1, x_2, x_3 : if we multiply them respectively by r_1, r_2, r_2 and add them together we get

$b_1r_1.\sin.(A_3-A_2+x_1)+b_2r_2.\sin.(A_1-A_3+x_2)+b_3r_3.\sin.(A_2-A_1+x_3)=0$

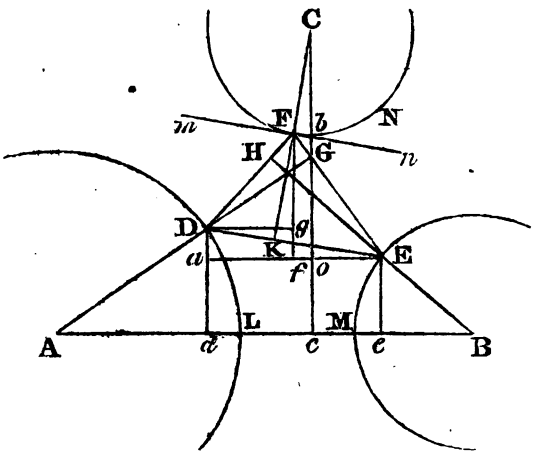
If the above three equations should be developed by means of tangents &c., it would seem as if the result would be so constituted that it would be better to assume *near* values of x_1, x_2, x_3, differing from the true values by the small quantities δx_1, δx_2, δx_3, and then by developing get a linear equation in δx_1, δx_2, δx_3 neglecting higher powers.

SECOND PRIZE SOLUTION.—*By Dr. Adrain.*

Let A, B, C be the centres of the given circles, and DEF the least triangle having its angular points on their circumferences in D, E, F.

Supposing D and E to be fixed while F varies its position, it is obvious that the triangle DEF will be least when the perpendicular FK on the base DE is least that is when the tangent MFN at F is parallel to DE, that is when KF produced passes through the centre C ; from which it appears

that the three radii AD, BE, GF produced meet the three sides of the triangle DEF at right angles in G, H, and K. The same observation is equally applicable to the case of the maximum, it will therefore be sufficient to give the solution for the minimum.



Draw Dd, Cc, Ee, Ef at right angles to AB, and Ea, Dg, Fb parallel to it; and from the preceding observation it is evident that the triangles ADd, EEf are similar; as also BEe, DFg; and GFb, EDa; if therefore we express the bases and perpendiculars of these right angled triangles by means of three unknown quantities, three equations will be obtained by the equalities of the three sets of tangents.

Put $AB=a'$, $AC=a''$, $Cc=b''$, $AD=r$, $BE=r'$, $CF=r''$; all which are given, and let A, B, G denote the unknown angles at the centres A, B, C; and the three equations expressing the equalities of tangents, are

I. $\dfrac{a'-a''-r\cos.B+r''\sin.C}{b''-r\sin.B-r''\cos.C}=\dfrac{\sin.A}{\cos.A}$; II. $\dfrac{a''-r\cos.A-r''\sin.C}{b''-r\sin.A-r''\cos.C}=\dfrac{\sin.B}{\cos.B}$; III. $\dfrac{r\sin.A-r'\sin.B}{a'-r\cos.A-r'\cos.B}=\dfrac{\sin.C}{\cos.C}$.

Clear equations I, and II, of fractions, and we have

$r''\{\sin.A\cos.C+\cos.A\sin.C\}=b''\sin.A-(a'-a'')\cos.A+r\cos.(A-B)=M$,

$r''\{\sin.B\cos.C-\cos.B\sin.C\}=b''\sin.B-a''\cos.B+r\cos.(A+B)=N$.

Using M and N for brevity instead of their values, we have by the common rules for simple equations,

$$\sin.\ c=\frac{M\sin.\ B-N\sin.\ A}{r''.\sin.\ (A+B)},\quad \cos.\ c=\frac{M\cos.\ B+N\cos.\ A}{r''.\sin.\ (A+B)}$$

Now since $\sin.^2c+\cos.^2c=1$ and that the value of $\frac{\sin.\ c}{\cos.\ c}$ is given by equation III. we have the two equations

$$(1)\quad M^2+N^2+2MN\cos.\ (A+B)=r''^2.\sin.^2(+B),$$

$$\frac{M\sin.\ B-N\sin.\ A}{M\cos.\ B+N\cos.A}=\frac{r\sin.\ A-r'\sin.\ B}{a'-r\cos.A-r'\cos.\ B}$$

which determine the values of A and B for the minimum. When near values of A and B are known which may easily be obtained by graphical operations; more accurate values may be obtained by any of the common methods of approximation.

THIRD PRIZE SOLUTION.—*By Mr Eugenius Nulty.*

Let the equations of the circles referred to rectangular axes be

$$(a-x)^2+(b-y)^2=c^2 \qquad (1)$$
$$(a'-x')^2+(b'-y')^2=c'^2 \qquad (2)$$
$$(a''-x'')^2+(b''-y'')^2=c''^2 \qquad (3)$$

The area of the triangle joining the points (x, y), (x', y'), (x'', y''), is

$$\tfrac{1}{2}\{(xy'-x'y)+(x'y''+x''y')+(x''y-xy'')\}$$

the differential of which corresponding to the case of maximum or minimum, gives by virtue of the preceding equations differentiated,

$$\frac{y'-y}{x'-x}=-\frac{a-x}{b-y} \qquad (4)$$

$$\frac{y''-y}{x''-x}=-\frac{a'-x'}{b'-y'} \qquad (5)$$

$$\frac{y-y'}{x-x'}=-\frac{a''-y''}{b''-y''} \qquad (6)$$

from which it appears that the radii joining the angular points of the required triangle, and the centres of the given circles are respectively perpendicular to the sides of the triangle, and consequently that the radii produced intersect each other in the same point.

The equations (4) and (5) give

$$x''=\frac{x'(a-x).(b'-y')-x(a'-x').(b-y)+(y'-y).(b-y).(b'-y')}{(a-x).(b'-y')-(a'-x').(b-y)}$$

$$y''=\frac{y(a-x).(b'-y')-y'(a'-x').(b-y)+(x-x').(b-x).(a'-x')}{(a-x).(b'-y')-(a'-x').(b-y)}$$

and assuming

$$a-x=c.\frac{1-r^2}{1+r^2},\ b-y=c.\frac{2r}{1+r^2}$$

$$(a'-x')=c'.\frac{1-s}{1+s},\ b'-y'=c'.\frac{2s}{1+s^2},$$

the equations (1) and (2) will be satisfied, and by substitution the equations (3) and (6) will take the form

$$r^m(s^n+\&c.)+.\quad =\sigma$$

$$r^{m'}(s^{n'}+\&c.)+\quad =\sigma$$

which by elimination will lead to an equation of the form

$$r^{\mu'}+\alpha r^{\mu'}+\&c.\quad =\sigma$$

the degree μ of which corresponding to general values of c, c', c'' and of the distances between the centres of the given circles, may be as high as $(m+n).(m'+n')-mn'$, that is in the present case as high as $(8+8).(6+6)-8.6$ or 144.

Particular cases can be attended with no difficulty, nor do they lead to any property that could not be easily anticipated.

ACKNOWLEDGMENTS, &c.

The following gentlemen favoured the editor with solutions to all the questions in Art. XV. No. VII. The figures annexed to the names refer to the questions answered by each as numbered in that article.

As question XXI. appears to have been understood differently by different contributors, it has been reproposed in this number of the Diary with the omission of the resistances. Vide New Questions.

Dr. Adrain, Rutger's College; Dr. Bowditch, Boston; Mr. Eugenius Nulty, Philadelphia; Dr. Anderson, Columbia College; and Professor Strong, Hamilton College; each answered all the questions.

Mr. M. O'Shannessy, N. Y. answered all but 20; Omicron, S. C. all but 20, 23; Mr. Benjamin Pierce, Jun. Cambridge College, Mass., all but 16, 19, 20, 22, 23; Mr. James Macully, N. Y. all but 9, 16, 18, 19, 20, 23; Mr. Nathan Brown, Jun'r., all but 13, 14, 15, 18, 19, 20, 22, 23; Charles Potts, Phil. all but 13, 14, 15, 16, 18, 20, 22; Mr. Gerardus B. Docharty, answered 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 16, 17; Mr. Silas Warner, Wrightstown, Penn. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 17; Mr. James Divver, S. C. Columbia, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 17; Mr. William J. Lewis, Phil. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 17; Mr. John D. Williams, N. Y. 1, 2, 3, 4, 5, 6, 7, 10, 12, 13, 16, 17, 22; Mr. John Swinburne,

Brooklyn, 2, 3, 4, 5, 6, 7, 8, 10, 11, 17; Mr. Thomas J. Megear, Wilmington, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 17; Mr. James Sloan, Middletown, N. J. 2, 3, 4, 5, 6, 7, 10, 11, 12; Mr. Henry Darnall, Phil. 2, 3, 4, 5, 7, 10, 15, 17; Mr. John B. Moreau, N. Y. 1, 2, 3, 4, 5, 7, 10; Mr. James O'Farrell, N. Y. 1, 2, 3, 4, 5, 7; Mr. John Delafield, N. Y. 2, 3, 4, 5, 7; Mr. James Maginness, Jun'r., Harr. Penn. 2, 4, 5, 7; Mr, Robert Parry, Mullica Hill, 1, 2, 3; Mr. William Vogdes, 2, 3, 4; Mr. M. Nash, N. Y. 2, 13; Correspondent, Lexington Kentucky, 13, 14; Mr. Alpheus Bixby, N. Y. 12; Mr. B. Mc'Gowan, N. Y. 14; Nemo, N. Y. 15; Mr. Michael Floy, N. Y. 6; Mr. Farrell Ward, N. Y. 9; Mary Bond, 10; Mr. Edward Giddings, 11; Mr. William Lenhart, 12; Mr. Charles Wilder, 16; Correspondent, 22; Mr. John Smith, Prize Question.

Having found it impossible to give the preference to any one of the solutions to the Prize Question by Dr. Bowditch, Dr. Adrain, and Mr. Eugenius Nulty, the editor has published all as prize solutions, and the *Prize* has therefore been divided equally among the above named gentlemen.

ARTICLE XVII.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN No. IX.

QUESTION I. (131).—*By Mr. William Vogdes.*

Given $xy+\frac{y^3}{x}=40$, and $\frac{x^3}{y}-xy=96$, to find the values of x and y by a quadratic.

QUESTION II. (132).—*By the same.*

What is the area of a right angled triangle whose sides are in arithmetical proportion; the least side squared and divided by 6, that quotient multiplied by the mean difference of the sides, will be equal to the area.

QUESTION III. (133).—*By Mr. John Swinburne.*

Given the sum of two perpendiculars drawn from a point in the side of an equilateral triangle upon the other two sides; to determine the triangle.

QUESTION IV. (134.)—*By Mr. John B. Moreau. N. Y.*

Five boys, A, B, C, D, and E, set out at the same time to run round a park, the circumference of which is 500

yards, A goes 124 yards in a minute, B 119, C 117, D 113, and E 107. I demand in what time will they all come together again; and how many times will A and E meet?

QUESTION V. (135).—*By the same.*

Given $\left\{\begin{array}{l} x(\sqrt{y}+1)+2\sqrt{xy}=55-y(\sqrt{x}+1) \\ \text{and } x\sqrt{y}+y\sqrt{x}=30. \end{array}\right\}$ to find the values of x and y.

QUESTION. VI. (136.)—*By Mr. George Alsop, Phil.*

In a given paraboloid it is required to describe the greatest cylinder.

QUESTION VII. (137.)—*By Mr. Edward Giddings.*

Given the vertical angle, the ratio of the sides, and the radius of the inscribed circle, to construct the triangle.

QUESTION VIII. (138.)—*By Mr. William Lenhart, York, Penn.*

If on a given base (286) of a plane triangle, between two acute angles, a semicircle be described, the circumference of which cuts the other two sides; and there be given the straight line joining the points of intersection 110, and the straight line bisecting the vertical angle and terminating in said line 68. It is required to determine the triangle.

QUESTION IX. (139.)—*By Mr. John Delafield Jun.*

Being called upon to survey a rectangular piece of ground whose length I found to be double the breadth, and that the length of a line, in feet, drawn from the top of a perpendicular pole, placed in one of the corners, (whose height was equal to one fourth of the breadth,) was equal to the area in square perches; to determine the sides and area of the triangle.

QUESTION X. (140.)—*By Diophantus.*

To find two numbers whose sum shall be a cube, and the sum of their squares increased by thrice their sum shall be a square.

QUESTION XI. (141.)—*By the same.*

To find two numbers, whose sum shall be an integral cube, and such that the square of each number increased by the other shall be an integral square.

QUESTION XII. (142.)—*By Philomath, Winchester, Va.*

Required the dimensions of a right angled triangle, the hypothenuse of which is given; the perpendicular being added to twice the base being a maximum.

QUESTION XIII. (143.)—*By Mr. Silas Warner.*

It is required to inscribe in a given hyperbola, a rectangle whose area, shall be a maximum.

QUESTION XIV. (144.)—*By Mr. William J. Lewis.*

A circle and its tangent being given in position and magnitude, it is required to draw a line from the extremity of the tangent cutting the circle so that the part of the line intercepted by the circle shall be equal to the tangent.

QUESTION XV. (145.)—*By Mr. Nathan Brown.*

The angular points of an acute angled triangle are the centres of three circles, each of which cuts off half the area of the triangle. Given the radii of these circles to determine the triangle whose angular points are made by the intersection of the circles with one another.

QUESTION XVI. (146.)—*By Mr. Thomas J. Megear.*

On a given base it is required to construct a right angled triangle such that a perpendicular drawn from the right angle to the hypothenuse shall cut off the greater segment of it equal to the remaining side of the triangle.

QUESTION XVII. (147.)—*By Mr. James Macully, N. Y.*

It is required to inscribe in a given paraboloid the greatest possible cone the diamete of whose base shall be perpendicular to the base of the paraboloid.

QUESTION XVIII. (148.)—*By* Ομικρον, *N. C.*

Given the difference of the lengths of a shadow on two given days at noon, to determine the latitude.

QUESTION XIX. (149.)—*By M. O'Shannessy, A. M.*
Required to find the content of the least sphere that shall touch any three right lines given in space.

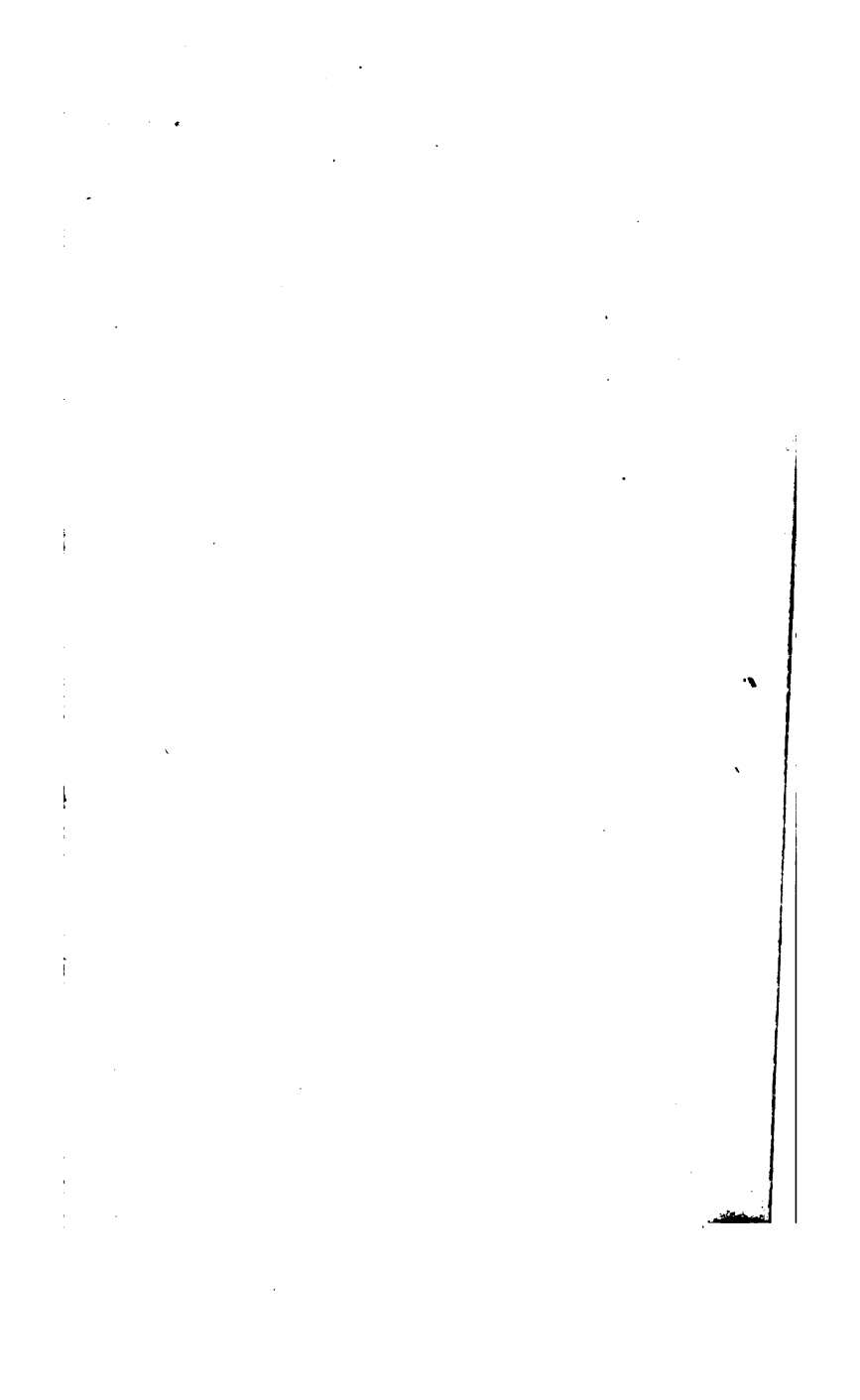
QUESTION XX. (150)—*By Mr. Matthew Collins, Professor of Mathematics, Limerick, Ireland.*
Given the hypothenuse of a right angled triangle, to construct it when the product of the *n*th power of the bisecter of one acute angle, and the *m*th power of the segment of the other leg between the bisecter and the other acute angle, is a maximum, *m* and *n*, bring any integers, and show previously a strictly geometrical analysis of every step of the composition.

QUESTION XXI. (151.)—*By Professor Strong.*
Two trees of given heights, stand on a side hill, which is an inclined plane having a given inclination to the horizon, a person whose eye is situated in the given plane at a certain point measures the angles subtended by the trees; from these data it is required to find the place of the observer's eye, supposing the line joining the foot of the trees is also given in length and position.

QUESTION XXII. (152.)—*By Mr. J. H. Swale, Liverpool, Eng.*
To constitute a triangle, having one extremity of its base at a point given in position, and the other angular points posited on the periphrasis of two circles, given in position and magnitude: when the base and the vertical angle, or the base and the sum of the squares of the sides; are either given, or a maximum or minimum.

QUESTION XXIII. (153.)—OR PRIZE QUESTION, *by Dr. Anderson.*
To determine the motion of a uniform heavy inflexible circular plate, placed originally in a position nearly vertical upon a horizontal plane, and then impelled in the direction of its plane; supposing the friction just sufficient to make the plates' circumference tend to roll without sliding, along its variable projection on the horizontal plane.

END OF VOL. I.



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